

How to Gauge Accuracy of Measurements and of Expert Estimates: Beyond Normal Distributions

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Need to Gauge Accuracy

Normally Distributed ...

Need to Go Beyond ...

To Solve the Problem, ...

Is It Possible to ...

Conclusion

Practical Question: ...

Resulting Algorithm

Home Page

Title Page

⏪

⏩

◀

▶

Page 1 of 12

Go Back

Full Screen

Close

Quit

1. Need to Gauge Accuracy

- The results \tilde{x} of measuring of estimating a quantity x are never absolutely accurate: $\Delta x \stackrel{\text{def}}{=} \tilde{x} - x \neq 0$.
- *It is important* to know the accuracy Δx .
- *Sometimes*, we do not know this accuracy; then we must estimate it based on the data itself.
- *Usually*, we compare \tilde{x} with the result \tilde{x}_s of meas. x w/a more accurate (“standard”) measuring instrument:

$$\text{since } \tilde{x}_s \approx x, \text{ we have } \Delta x \approx \tilde{x} - \tilde{x}_s.$$

- *Problem:* how to calibrate state-of-the-art measurements? examples:
 - in the environmental sciences, we measure Carbon and heat fluxes via Eddy covariance towers;
 - in geosciences, we use seismic, gravity, etc., to find density at different depths and locations.

2. Normally Distributed Measurement Errors with Mean 0 and Unknown Variance V

- If we have two similar MIs, then $\tilde{x}^{(1)} - \tilde{x}^{(2)} = \Delta x^{(1)} - \Delta x^{(2)}$ is normally distributed w/variance $V' = 2V$.
- From the sample of differences, we estimate V' and estimate V as $V'/2$.
- *Example:* two nearby Eddy Covariance towers.
- In geosciences, we usually have only one seismic map, only one gravity map, etc.
- In general, we have several measurement results $\tilde{x}^{(i)}$ with variances V_i .
- Here, the variance e_{ij} of the difference $\tilde{x}^{(i)} - \tilde{x}^{(j)}$ is equal to $e_{ij} = V_i + V_j$.
- We have 3 equations $e_{12} = \dots$, $e_{23} = \dots$, $e_{13} = \dots$ for 3 unknown variances, so, e.g., $V_1 = \frac{e_{12} + e_{13} - e_{23}}{2}$.

Need to Gauge Accuracy

Normally Distributed ...

Need to Go Beyond ...

To Solve the Problem, ...

Is It Possible to ...

Conclusion

Practical Question: ...

Resulting Algorithm

Home Page

Title Page



Page 3 of 12

Go Back

Full Screen

Close

Quit

3. Need to Go Beyond Normal Distributions, and Resulting Problem

- The distribution of measurement errors is sometimes *not normal* (e.g., in measuring fluxes).
- *In such cases*, in addition to variance V , we need to know skewness and other characteristics.
- *In general*, reconstruction of an asymmetric distribution is not unique.
- *Proof*: Δx and $\Delta y = -\Delta x$ lead to the same distribution for differences $\tilde{x}^{(1)} - \tilde{x}^{(2)} = \Delta x^{(1)} - \Delta x^{(2)}$.
- *Natural questions*:
 - which characteristics of the distribution Δx *can* we reconstruct?
 - what are efficient *algorithms* for this reconstruction?

Need to Gauge Accuracy

Normally Distributed ...

Need to Go Beyond ...

To Solve the Problem, ...

Is It Possible to ...

Conclusion

Practical Question: ...

Resulting Algorithm

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 4 of 12

Go Back

Full Screen

Close

Quit

4. To Solve the Problem, We Use Fourier Analysis

- We want to find is the probability density $\rho(z)$ describing the distribution of the measurement error $z \stackrel{\text{def}}{=} \Delta x$.
- In order to find the unknown probability density, we will first find its Fourier transform

$$F(\omega) = \int \rho(z) \cdot e^{i\omega \cdot z} dz = E [e^{i\omega \cdot z}].$$

- Such a mathematical expectation is also known as a *characteristic function* of the random variable z .
- Based on the observed values of the difference $z^{(1)} - z^{(2)}$, we can estimate its characteristic function

$$D(\omega) = E [e^{i\omega \cdot (z^{(1)} - z^{(2)})}].$$

- It is known that $D(\omega) = F(\omega) \cdot F^*(\omega) = |F(\omega)|^2$.
- How can we reconstruct the complex-valued function $F(\omega)$ if we only know its absolute value?

Need to Gauge Accuracy

Normally Distributed ...

Need to Go Beyond ...

To Solve the Problem, ...

Is It Possible to ...

Conclusion

Practical Question: ...

Resulting Algorithm

Home Page

Title Page



Page 5 of 12

Go Back

Full Screen

Close

Quit

5. Is It Possible to Estimate Accuracy?

- Theoretically, we can consider all possible values of the difference $z^{(1)} - z^{(2)}$.
- In practice, we can only get values proportional to the smallest measuring unit h (e.g., $h = 1$ cm).
- In the 1-D case, the Fourier transform takes the form
$$F(\omega) = \sum_{k=0}^N p_k \cdot s^k, \text{ where } s \stackrel{\text{def}}{=} e^{i\omega \cdot h}.$$
- In the multi-D case, we have $z = (k_1 \cdot h_1, k_2 \cdot h_2, \dots)$, and
$$F(\omega) = \sum_{k_1=0}^{N_1} \sum_{k_2=0}^{N_2} \dots p_{k_1, k_2, \dots} \cdot s_1^{k_1} \cdot s_2^{k_2} \cdot \dots$$
- In terms of polynomials, the question takes the following form:
 - we know the values $D(s) = |P(s)|^2 = P(s) \cdot P^*(s)$ for some polynomial $P(s)$,
 - we need to reconstruct this polynomial $P(s)$.

Need to Gauge Accuracy

Normally Distributed ...

Need to Go Beyond ...

To Solve the Problem, ...

Is It Possible to ...

Conclusion

Practical Question: ...

Resulting Algorithm

Home Page

Title Page

◀

▶

◀

▶

Page 6 of 12

Go Back

Full Screen

Close

Quit

6. Is It Possible to Estimate Accuracy (cont-d)

- In 1-D case, each complex-valued polynomial of degree N has, in general, N complex roots $s^{(1)}$, $s^{(2)}$, etc.
- Thus, $P(s) = \text{const} \cdot (s - s^{(1)}) \cdot (s - s^{(2)}) \cdot \dots$ and $|P(s)|^2 = \text{const} \cdot (s - s^{(1)}) \cdot (s - s^{(1)})^* \cdot \dots$
- There are many factors, so there are many ways to represent it as a product – reconstruction is not unique.
- In the multi-D case, a generic polynomial *cannot* be represented as a product of polynomials.
- E.g., to describe a polynomial $\sum_{k=0}^n \sum_{l=1}^n c_{kl} \cdot s_1^k \cdot s_2^l$ of degree n , we need $(n + 1)^2$ coefficients.
- When polyn. multiply, degrees add: $s^m \cdot s^{m'} = s^{m+m'}$.
- Thus, if $P(s)$ is a product of two polynomials, one has a degree $m < n$, and the other degree $n - m$.

Need to Gauge Accuracy

Normally Distributed ...

Need to Go Beyond ...

To Solve the Problem, ...

Is It Possible to ...

Conclusion

Practical Question: ...

Resulting Algorithm

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 7 of 12

Go Back

Full Screen

Close

Quit

7. Proof (cont-d)

- If $P(s)$ is a product of two polynomials, one has a degree $m < n$, and the other degree $n - m$.
- In general:
 - we need $(m + 1)^2$ coefficients to describe a polynomial of degree m and
 - we need $(n - m + 1)^2$ coefficients to describe a polynomial of degree $n - m$,
 - so to describe arbitrary products of such polynomials, we need $(m + 1)^2 + (n - m + 1)^2$ coefficients.
- In general, the total number of coefficients is smaller than $(n + 1)^2$.
- So, a general polynomial cannot be represented as a product of two polynomials.

Need to Gauge Accuracy

Normally Distributed ...

Need to Go Beyond ...

To Solve the Problem, ...

Is It Possible to ...

Conclusion

Practical Question: ...

Resulting Algorithm

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 8 of 12

Go Back

Full Screen

Close

Quit

8. Conclusion

- We have shown that a general polynomial cannot be represented as a product of two polynomials.
- Thus, $D(s) = P(s) \cdot P^*(s) = Q(s) \cdot Q^*(s)$ implies that $Q(s) = P(s)$ or $Q(s) = P^*(s)$.
- In the first case, we get $\rho(x)$.
- In the second case, we get $\rho(-x)$.
- So, in general, only $\rho(x)$ and $\rho(-x)$ are consistent with the observed differences $\Delta x^{(1)} - \Delta x^{(2)}$.
- Thus, we *can* reconstruct the distribution $\rho(x)$ of measurement errors – modulo $x \rightarrow -x$.

Need to Gauge Accuracy

Normally Distributed ...

Need to Go Beyond ...

To Solve the Problem, ...

Is It Possible to ...

Conclusion

Practical Question: ...

Resulting Algorithm

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 9 of 12

Go Back

Full Screen

Close

Quit

9. Practical Question: How to Gauge the Accuracy

- *Problem:* find a function $\rho(z)$ which satisfies the following two conditions:
 - $\rho(z) \geq 0$ for all z , and
 - $|F(\omega)|^2 = D(\omega)$ for given $D(\omega)$.
- *Method:* of successive projections.
- We start with an arbitrary function $\rho^{(0)}(z)$.
- On the k -th iteration, starting with the result $\rho^{(k-1)}(z)$ of the previous iteration, we:
 - find the closest function $\rho'(x)$ to $\rho^{(k-1)}(z)$ which satisfies the 1st condition;
 - then, find the closest function $\rho^{(k)}(x)$ to $\rho'(z)$ which satisfies the 2nd condition.
- We continue this process until it converges.

Need to Gauge Accuracy

Normally Distributed ...

Need to Go Beyond ...

To Solve the Problem, ...

Is It Possible to ...

Conclusion

Practical Question: ...

Resulting Algorithm

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 10 of 12

Go Back

Full Screen

Close

Quit

10. Resulting Algorithm

- We start with an arbitrary function $\rho^{(0)}(z)$.
- On the k -th iteration, we start with the function $\rho^{(k-1)}(z)$ obtained on the previous iteration, and we:
 - first, we compute $\rho'(z) = \max(0, \rho^{(k-1)}(z))$;
 - then, we apply Fourier transform to $\rho'(z)$ and get $F'(\omega)$;
 - after that, we compute $F^{(k)}(\omega) = \frac{\sqrt{|D(\omega)|}}{|F'(\omega)|} \cdot F'(\omega)$;
 - as the next approx. $\rho^{(k)}(z)$, we take the result of applying the inverse Fourier transform to $F^{(k)}(\omega)$.
- We continue this process until it converges; this enables us to recover many $\rho(x)$.

Need to Gauge Accuracy

Normally Distributed ...

Need to Go Beyond ...

To Solve the Problem, ...

Is It Possible to ...

Conclusion

Practical Question: ...

Resulting Algorithm

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 11 of 12

Go Back

Full Screen

Close

Quit

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Need to Gauge Accuracy

Normally Distributed . . .

Need to Go Beyond . . .

To Solve the Problem, . . .

Is It Possible to . . .

Conclusion

Practical Question: . . .

Resulting Algorithm

[Home Page](#)

[Title Page](#)



Page 12 of 12

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)