Towards More Adequate Representation of Uncertainty: From Intervals to Set Intervals, with the Possible Addition of Probabilities and Certainty Degrees

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1. Need for Set Intervals

- *Ideal case:* complete knowledge.
- *We are interested in:* properties P_i such as "high fever", "headache", etc.
- *Complete:* we know the exact set S_i of all the objects that satisfy each property P_i.
- *In practice*, we usually only have *partial* knowledge:
  - the set $\underline{S}_i$ of all the objects about which we know that P_i holds, and
  - the set $\overline{S}_i$ about which we know that P_i *may* hold (i.e., equivalently, that we have not yet excluded the possibility of P_i).
- *Set interval:* the only information about the actual (unknown) set $S_i = \{x : P_i(x)\}$ is that $\underline{S}_i \subseteq S_i \subseteq \overline{S}_i$, i.e., that

$$S_i \in \mathbf{S}_i = [\underline{S}_i, \overline{S}_i] \stackrel{\text{def}}{=} \{S_i : \underline{S}_i \subseteq S_i \subseteq \overline{S}_i\}.$$

2. Need for Set Operations with Set Intervals

- *Main problem:*
  - we have some information about the original properties P_i;
  - we would like to describe the set $S = \{x : P(x)\}$ of all the values that satisfy some combination $P \stackrel{\text{def}}{=} f(P_1, \ldots, P_n)$.
- *Example (informal):* flu $\leftrightarrow$ high fever *and* headache *and not* rash.
- *Example (formal):* $f(P_1, P_2, P_3) = P_1 \,\&\, P_2 \,\&\, \neg P_3$.
- *Ideal case:* we know the exact sets $S_i = \{x : P_i(x)\}$.
- *Solution:*
  - $f(S_1, \ldots, S_n)$ is composition of union, intersection, and complement;
  - apply the corresponding set operation, step-by-step, to the known sets S_i.
- *General case:* describe the class $\mathcal{S}$ of all possible sets S corresponding to different $S_i \in \mathbf{S}_i$:

$$\mathcal{S} \stackrel{\text{def}}{=} \{f(S_1, \ldots, S_n) : S_1 \in \mathbf{S}_1, \ldots, S_n \in \mathbf{S}_n\}.$$

3. Elementary Set Operations and Their Use

- *Simplest case:* $n = 2$ and $f(P_1, P_2)$ is an elementary set operation (union, intersection, complement).
- *Useful property:* elementary set operations are monotonic in $\subseteq$.
- For these operations, formulas for estimating $\mathbf{S}$ are known:

$$[\underline{A}, \overline{A}] \cup [\underline{B}, \overline{B}] = [\underline{A} \cup \underline{B}, \overline{A} \cup \overline{B}]; \;\; [\underline{A}, \overline{A}] \cap [\underline{B}, \overline{B}] = [\underline{A} \cap \underline{B}, \overline{A} \cap \overline{B}];$$

$$-[\underline{A}, \overline{A}] = [-\overline{A}, -\underline{A}].$$

- *General case: idea* (similar to interval computations)
  - parse the expression $f(S_1, \ldots, S_n)$;
  - replace each elementary set operation by the corresponding operation with interval sets.
- *Result:* we get an enclosure for $\mathbf{S} = [\underline{S}, \overline{S}]$.
- *Problem:* we may get excess width.
- *Example:* for $f(S_1) = S_1 \cup -S_1$, $S_1 = [\emptyset, U]$.
  - *actual range:* $\mathbf{S} = \{U\}$;
  - *enclosure:* $-\mathbf{S_1} = [\emptyset, U]$, so $\mathbf{S_1} \cup -\mathbf{S}_1 = [\emptyset, U] \cup [\emptyset, U] = [\emptyset, U]$.

4. How to Get Exact Set Range? How Difficult Is It?

- *Problem:* in general, set operations such as $S_1 \cup -S_1$ are not $\subseteq$-monotonic.
- *Solution for computing $\overline{S}$:*
  - represent $f(S_1, \ldots, S_n)$ in a canonical DNF form
  $$(S_1 \cap -S_2 \cap \ldots \cap S_n) \cup (\ldots) \cup \ldots$$
  - apply straightforward interval computations:
  $$(\overline{S}_1 \cap -\underline{S}_2 \cap \ldots \cap \overline{S}_n) \cup (\ldots) \cup \ldots$$
- *Proof:* each element from each conjunction $\overline{S}_1 \cap -\underline{S}_2 \cap \ldots \cap \overline{S}_n$ is possible.
- *Example:* $S_1 \bigtriangleup S_2 = (S_1 \cap -S_2) \cup (-S_1 \cap S_2))$, so
$$\overline{S} = (\overline{S}_1 \cap -\underline{S}_2) \cup (-\underline{S}_1 \cap \overline{S}_2).$$
- *Solution for computing $\underline{S}$:* use $\underline{S} = -(\overline{-S})$, i.e., use CNF.
- *Problem:* turning into DNF or CNF requires exponential time.
- *Comment:* the problem of checking whether $\emptyset \in f(\mathbf{S}_1, \ldots, \mathbf{S}_n)$ is NP-hard.

5. Intermediate Value Theorem for Set Intervals

- *Situation:* in the range $\mathcal{S} = f(\mathbf{S}_1, \ldots, \mathbf{S}_n)$, we found the intersection $\underline{S}$ and the union $\overline{S}$ of all possible sets.
- *Conclusion:* $\mathcal{S} \subseteq [\underline{S}, \overline{S}]$.
- *Theorem:* $\mathcal{S} = [\underline{S}, \overline{S}]$.
- *Equivalent formulation:* for every $S \in [\underline{S}, \overline{S}]$, there exist sets

$$S_1 \in [\underline{S}_1, \overline{S}_1], \ldots, S_n \in [\underline{S}_n, \overline{S}_n]$$

  for which $S = f(S_1, \ldots, S_n)$.
- *Difficulty:* values $S_i(u)$ and $S(u)$ are discrete (0 or 1), so the standard intermediate value theorem does not apply.
- *Solution:* we define S_i element-by-element.
- *Known:* for each $u \in U$, we have $\underline{S}(u) \le S(u) \le \overline{S}(u)$.
- *Conclusion:* $S(u) = \underline{S}(u)$ or $S(u) = \overline{S}(u)$.
- *By definition* of $\underline{S}$ and $\overline{S}$, in both cases, there exist sets $s_i^{(u)}$ for which $S(u) = f(s_1^{(u)}(u), \ldots, s_n^{(u)}(u))$.
- We take $S_i(u) = s_i^{(u)}(u)$.

6. Fuzzy Sets

- *Previous description:*
  - about some elements u, we know $P(u)$;
  - about some elements u, we know $\neg P(u)$:
  - about other elements u, we know nothing about $P(u)$.
- *Description:* sets $\underline{S}$ and $\underline{(-S)} = -\overline{S}$.
- *Additional information:* experts may believe that $P(u)$ holds with some certainty α.
- *How to describe this information:* a nested family of sets S_α corresponding to α:
  - $S_0 = \overline{S}$;
  - $S_1 = \underline{S}$;
  - if $\alpha < \alpha'$ then $S_\alpha \subseteq S_{\alpha'}$.
- *Traditional description:* $\mu_A(u) = \max\{\alpha : u \in S_\alpha\}$.
- *Set operations in terms of μ:* $\mu_{A\cup B}(u) = \max(\mu_A(u), \mu_B(u))$; $\mu_{A\cap B}(u) = \min(\mu_A(u), \mu_B(u))$; $\mu_{\neg A}(u) = 1 - \mu_A(u)$.

7. Interval-Valued Fuzzy Sets

- *Situation:* for every α, we are not sure which elements belong to S_α and which do not.
- *Description:* $\underline{S}_\alpha \subseteq \overline{S}_\alpha$.
- *Alternative description:* interval-valued membership function

$$[\underline{\mu}_A(u), \overline{\mu}_A(u)].$$

- *Meaning:* for all u, we have $\mu_A(u) \in [\underline{\mu}_A(u), \overline{\mu}_A(u)]$, i.e.,

$$\underline{A} \subseteq A \subseteq \overline{A}.$$

- *Problem:*
  - we know $\mathbf{A}_1, \ldots, \mathbf{A}_n$,
  - we know that $A = f(A_1, \ldots, A_n)$ for some set-expression f;
  - find the range of A:

$$f(\mathbf{A}_1, \ldots, \mathbf{A}_n) = \{f(A_1, \ldots, A_n) : A_1 \in \mathbf{A}_1, \ldots, A_n \in \mathbf{A}_n\}.$$

8. Solution

- *Negative result:* in general, the problem is NP-hard.
- *Straightforward interval computations:*

$$[\underline{\mu}_A(u), \overline{\mu}_A(u)] \cup [\underline{\mu}_B(u), \overline{\mu}_B(u)] = [\max(\underline{\mu}_A(u), \underline{\mu}_B(u)), \max(\overline{\mu}_A(u), \overline{\mu}_B(u))];$$

$$[\underline{\mu}_A(u), \overline{\mu}_A(u)] \cap [\underline{\mu}_B(u), \overline{\mu}_B(u)] = [\min(\underline{\mu}_A(u), \underline{\mu}_B(u)), \min(\overline{\mu}_A(u), \overline{\mu}_B(u))];$$

$$-[\underline{\mu}_A(u), \overline{\mu}_A(u)] = [1 - \overline{\mu}_A(u), 1 - \underline{\mu}_A(u))].$$

- *Good news:* we always get an enclosure.
- *Bad news:* excess width.
- *Solution: idea.* Use DNF for $\overline{A}$ and CNF for $\underline{A}$.
- *Details:* it is slightly different from the usual since we view P and $\neg P$ as separate literals.
- Here, $A \cap -A$ is not transformed into $\emptyset$, so we may have

$$(A_1 \cap -A_1 \cap A_2 \cap -A_3 \ldots) \cup (\ldots) \ldots$$

- *Intermediate value theorem:* follows from continuity of element-by-element function $A(u) = f(A_1(u), \ldots, A_n(u))$.

9. Probabilistic Case: In Brief

- *Situation:* we know $p(A_i)$, we want estimates for $p(A)$, where $A = f(A_1, \ldots, A_n)$.
- *In general:* NP-hard.
- *Exp-time algorithm:* LP with $p(A_1 \,\&\, -A_2 \,\&\, \ldots)$ etc.
- *Feasible algorithm:* expert systems use technique similar to straightforward interval computations.
- *Details:* we parse F and replace each computation step with corresponding probability operation.
- *Problem:* at each step, we ignore the dependence between the intermediate results F_j.
- *Result:* intervals are too wide (and numerical estimates off).
- *Example:* the estimate for $P(A \vee \neg A)$ is not 1.
- *Solution:* similarly to the above algorithm, besides $P(F_j)$, we also compute $P(F_j \,\&\, F_i)$ (or $P(F_{j_1} \,\&\, \ldots \,\&\, F_{j_k})$).
- On each step, use all combinations of l such probabilities to get new estimates.
- *Result:* e.g., $P(A \vee \neg A)$ is estimated as 1.

10. Similar Idea for Sets

- *Problem:* estimate the range of $f(S_1, \ldots, S_n)$ in polynomial time.
- *Previous algorithm:* for each intermediate set $S_m = S_i \oplus S_j$, we use bounds on S_i and S_j to find bounds on S_m.
- *New idea:* for each m, in addition to bounds on S_m, we also keep (and compute) bounds on

$$S_{m,k} \stackrel{\text{def}}{=} S_m \cap S_k, \quad S_{m,-k} \stackrel{\text{def}}{=} S_m \cap -S_k,$$

$$S_{-m,k} \stackrel{\text{def}}{=} -S_m \cap S_k, \quad S_{-m,-k} \stackrel{\text{def}}{=} -S_m \cap -S_k,$$

  for all $k \le n$.
- *Example:* $S_m = S_i \cap S_j$, then

$$S_m \cap S_k = (S_i \cap S_k) \cap (S_j \cap S_k) \text{ so } \overline{S}_{m,k} = \overline{S}_{i,k} \cap \overline{S}_{j,k};$$

$$S_m \cap -S_k = (S_i \cap -S_k) \cap (S_j \cap -S_k) \text{ so } \overline{S}_{m,-k} = \overline{S}_{i,-k} \cap \overline{S}_{j,-k};$$

$$-S_m \cap S_k = (-S_i \cap S_k) \cup (-S_j \cap S_k) \text{ so } \overline{S}_{m,k} = \overline{S}_{-i,k} \cup \overline{S}_{-j,k};$$

$$-S_m \cap -S_k = (-S_i \cap -S_k) \cup (-S_j \cap -S_k) \text{ so } \overline{S}_{m,k} = \overline{S}_{-i,-k} \cup \overline{S}_{-j,-k}.$$

- *Comment:* similar algorithm is possible for fuzzy sets.

11. Acknowledgments

This work was supported in part:

- by National Science Foundation grants EAR-0225670 and DMS-0532645 and
- by Texas Department of Transportation grant No. 0-5453