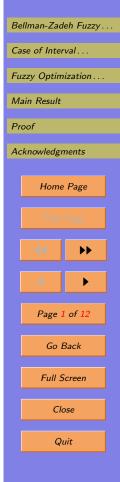
Bellman-Zadeh Fuzzy Optimization Under Interval Uncertainty

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1. Bellman-Zadeh Fuzzy Optimization

- In many real-life situations:
 - in addition to well-defined constraints that limit alternatives x to a certain set X,
 - we also have *fuzzy* constraints like "temperature should not be too high".
- For such constraints, we do not know exactly
 - which alternatives x satisfy the desired constraint and
 - which do not.
- Instead, we only have degree of confidence $\mu(x) \in [0, 1]$ that describes
 - to what extent the experts believe
 - that the alternative x satisfies the desired constraints.



2. Bellman-Zadeh Fuzzy Optimization (cont-d)

- Usually, we have an objective function f(x) that we want to maximize.
- How do we optimize it under such fuzzy constraints?
- A solution to this problem was proposed in a joint 1970 paper:
 - by Richard Bellman of optimization fame and
 - by Lotfi Zadeh, father of fuzzy techniques.
- First, we select an "and"-operation $f_{\&}(a, b)$ a function that is non-decreasing with respect to a and b.
- \bullet Then, we select an alternative x that maximizes the expression

$$F(x) \stackrel{\text{def}}{=} f_{\&} \left(\frac{f(x) - f_{-}}{f_{+} - f_{-}}, \mu(x) \right)$$
, where

$$f_{-}\stackrel{\mathrm{def}}{=}\inf\{f(y):y\in X\}$$
 and $f_{+}\stackrel{\mathrm{def}}{=}\sup\{f(y):y\in X\}.$

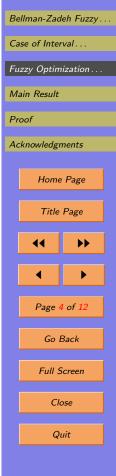
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3. Case of Interval Uncertainty

- In the ideal case:
 - we know the exact values of the objective function f(x), and
 - we know the exact values of the expert's degree of confidence $\mu(x)$.
- In practice, we often only know f(x) and $\mu(x)$ with interval uncertainty.
- In other words, for every x, we only know the bounds $f(x) \le f(x) \le \overline{f}(x)$ and $\mu(x) \le \mu(x) \le \overline{\mu}(x)$.



4. Fuzzy Optimization Under Interval Uncertainty: Formulation of the Problem

- For different $f(x) \in [\underline{f}(x), \overline{f}(x)]$ and $\mu(x) \in [\underline{\mu}(x), \overline{\mu}(x)]$, we get different values of F(x).
- So, to make a decision, it is reasonable to find the range $[\underline{F}(x), \overline{F}(x)]$ of possible values of F(x).
- Once we have found this range, we can select all the alternatives which can be optimal for some

$$F(x) \in [\underline{F}(x), \overline{F}(x)].$$

• This is equivalent to selecting all alternatives for which

$$\overline{F}(x) \ge \sup_{y} \underline{F}(y).$$



5. Formulation of the Problem (cont-d)

- If we want to select a single alternative, we can follow the usual Hurwicz decision-making strategy:
 - find the value $\alpha \in [0,1]$ that reflects the decision maker's degree of optimism-pessimism, and
 - select the alternative x that maximizes the expression

$$F_{\alpha}(x) \stackrel{\text{def}}{=} \alpha \cdot \overline{F}(x) + (1 - \alpha) \cdot \underline{F}(x).$$



6. Main Result

$$\underline{F}(x) = f_{\&} \left(\max \left(0, \frac{\underline{f}(x) - \underline{f}_{+}(x)}{\max(\underline{f}(x), \overline{f}_{+}(x)) - \underline{f}_{+}(x)} \right), \underline{\mu}(x) \right),$$

$$\operatorname{and} \overline{F}(x) =$$

$$f_{\&} \left(\min \left(1, \frac{\overline{f}(x) - \min(\overline{f}(x), \underline{f}_{-}(x))}{\max(\overline{f}(x), \overline{f}_{-}(x)) - \min(\overline{f}(x), \underline{f}_{-}(x))} \right), \overline{\mu}(x) \right),$$

$$\operatorname{where}$$

$$\underline{f}_{-}(x) \stackrel{\text{def}}{=} \inf\{\underline{f}(y) : y \in X, y \neq x\},$$

$$\underline{f}_{+}(x) \stackrel{\text{def}}{=} \sup\{\underline{f}(y) : y \in X, y \neq x\},$$

$$\overline{f}_{-}(x) \stackrel{\text{def}}{=} \inf\{\overline{f}(y) : y \in X, y \neq x\},$$

$$\overline{f}_{+}(x) \stackrel{\text{def}}{=} \sup\{\overline{f}(y) : y \in X, y \neq x\}.$$

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7. Proof

- Let us denote $f_{-}(x) \stackrel{\text{def}}{=} \inf\{f(y) : y \in X, y \neq x\}$ and $f_{+}(x) \stackrel{\text{def}}{=} \sup\{f(y) : y \in X, y \neq x\}.$
- Then, the formula for F(x) takes the form

$$F(x) = f_{\&} \left(\frac{f(x) - \min(f(x), f_{-}(x))}{\max(f(x), f_{+}(x)) - \min(f(x), f_{-}(x))}, \mu(x) \right).$$

- If $\underline{f}(x) \leq \overline{f}(x)$, then for $f(x) = \underline{f}(x)$ and $f(y) = \overline{f}(y)$ for all $y \neq x$, we have $f(x) \leq f(x)$.
- Hence $\min(f(x), f_{-}(x)) = f(x)$.
- So, $f(x) \min(f(x), f_{-}(x)) = 0$, and the ratio F(x) takes its smallest possible value 0.
- If $\underline{f}(x) > \overline{f}(x)$, this implies that f(x) > f(x) for all possible functions f, thus

$$F(x) = f_{\&} \left(\frac{f(x) - f_{-}(x)}{\max(f(x), f_{+}(x)) - f_{-}(x)}, \mu(x) \right).$$

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8. Proof (cont-d)

- This expression is (non-strictly) increasing in $\mu(x)$ and decreasing in $f_+(x)$.
- Thus its minimum is attained when:
 - $\mu(x)$ attains its smallest possible value $\mu(x)$ and
 - $f_+(x)$ attains its largest possible value $\overline{f_+}(x)$:

$$F(x) = f_{\&}\left(\frac{f(x) - f_{-}(x)}{\max(f(x), \overline{f_{+}}(x)) - f_{-}(x)}, \underline{\mu}(x)\right).$$

- If $\underline{f}(x) \ge \overline{f_+}(x)$, then $f(x) \ge \overline{f_+}(x)$, and thus, the ratio is always equal to its largest possible value 1.
- If $\underline{f}(x) < \overline{f_+}(x)$, then for $f(x) < \overline{f_+}(x)$, the ratio can be smaller than 1 and is equal to $\frac{f(x) f_-(x)}{\overline{f}(x) f_-(x)}$.

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9. Proof (cont-d)

- This expression is increasing with f(x).
- So its minimum is attained when f(x) attains its smallest possible value f(x), thus

$$F(x) = f_{\&} \left(\frac{\underline{f}(x) - f_{-}(x)}{\max(\underline{f}(x), \overline{f_{+}}(x)) - f_{-}(x)}, \underline{\mu}(x) \right).$$

• To find the dependence on $f_{-}(x)$, we can represent the ratio as

$$\frac{\underline{f}(x) - f_{-}(x)}{\max(\underline{f}(x), \overline{f_{+}}(x)) - f_{-}(x)} = 1 - \frac{\max(\underline{f}(x), \overline{f_{+}}(x)) - \underline{f}(x)}{\max(\underline{f}(x), \overline{f_{+}}(x)) - f_{-}(x)}.$$

- This expression is clearly decreasing in $f_{-}(x)$.
- So its minimum is attained when $f_{-}(x)$ attains its largest possible value $f_{-}(x) = \overline{f_{-}}(x)$.
- Thus, we get the desired formula for $\underline{F}(x)$.

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10. Proof (cont-d)

• Similar arguments explain the formula for $\overline{F}(x)$.



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