Bellman-Zadeh Fuzzy Optimization Under Interval Uncertainty

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1. Bellman-Zadeh Fuzzy Optimization

- In many real-life situations:
  - in addition to well-defined constraints that limit alternatives x to a certain set X,
  - we also have *fuzzy* constraints like "temperature should not be too high".
- For such constraints, we do not know exactly
  - which alternatives x satisfy the desired constraint and
  - which do not.
- Instead, we only have degree of confidence $\mu(x) \in [0, 1]$ that describes
  - to what extent the experts believe
  - that the alternative x satisfies the desired constraints.

2. Bellman-Zadeh Fuzzy Optimization (cont-d)

- Usually, we have an objective function $f(x)$ that we want to maximize.
- How do we optimize it under such fuzzy constraints?
- A solution to this problem was proposed in a joint 1970 paper:
  - by Richard Bellman of optimization fame and
  - by Lotfi Zadeh, father of fuzzy techniques.
- First, we select an "and"-operation $f_\&(a,b)$ – a function that is non-decreasing with respect to a and b.
- Then, we select an alternative x that maximizes the expression

$$F(x) \stackrel{\text{def}}{=} f_\&\left(\frac{f(x)-f_-}{f_+-f_-}, \mu(x)\right), \quad \text{where}$$

$$f_- \stackrel{\text{def}}{=} \inf\{f(y) : y \in X\} \text{ and } f_+ \stackrel{\text{def}}{=} \sup\{f(y) : y \in X\}.$$

3. Case of Interval Uncertainty

- In the ideal case:
  - we know the exact values of the objective function $f(x)$, and
  - we know the exact values of the expert's degree of confidence $\mu(x)$.
- In practice, we often only know $f(x)$ and $\mu(x)$ with interval uncertainty.
- In other words, for every x, we only know the bounds $\underline{f}(x) \le f(x) \le \overline{f}(x)$ and $\underline{\mu}(x) \le \mu(x) \le \overline{\mu}(x)$.

4. Fuzzy Optimization Under Interval Uncertainty: Formulation of the Problem

- For different $f(x) \in [\underline{f}(x), \overline{f}(x)]$ and $\mu(x) \in [\underline{\mu}(x), \overline{\mu}(x)]$, we get different values of $F(x)$.
- So, to make a decision, it is reasonable to find the range $[\underline{F}(x), \overline{F}(x)]$ of possible values of $F(x)$.
- Once we have found this range, we can select all the alternatives which can be optimal for some

$$F(x) \in [\underline{F}(x), \overline{F}(x)].$$

- This is equivalent to selecting all alternatives for which

$$\overline{F}(x) \geq \sup_y \underline{F}(y).$$

5. Formulation of the Problem (cont-d)

- If we want to select a single alternative, we can follow the usual Hurwicz decision-making strategy:
  - find the value $\alpha \in [0, 1]$ that reflects the decision maker's degree of optimism-pessimism, and
  - select the alternative x that maximizes the expression

$$F_\alpha(x) \stackrel{\text{def}}{=} \alpha \cdot \overline{F}(x) + (1 - \alpha) \cdot \underline{F}(x).$$

6. Main Result

$$\underline{F}(x) = f_\&\left(\max\left(0, \frac{\underline{f}(x) - \underline{f_+}(x)}{\max(\underline{f}(x), \overline{f_+}(x)) - \underline{f_+}(x)}\right), \underline{\mu}(x)\right),$$

and $\overline{F}(x) =$

$$f_\&\left(\min\left(1, \frac{\overline{f}(x) - \min(\overline{f}(x), \underline{f_-}(x))}{\max(\overline{f}(x), \overline{f_-}(x)) - \min(\overline{f}(x), \underline{f_-}(x))}\right), \overline{\mu}(x)\right),$$

where

$$\underline{f_-}(x) \stackrel{\text{def}}{=} \inf\{\underline{f}(y) : y \in X, y \neq x\},$$

$$\underline{f_+}(x) \stackrel{\text{def}}{=} \sup\{\underline{f}(y) : y \in X, y \neq x\},$$

$$\overline{f_-}(x) \stackrel{\text{def}}{=} \inf\{\overline{f}(y) : y \in X, y \neq x\},$$

$$\overline{f_+}(x) \stackrel{\text{def}}{=} \sup\{\overline{f}(y) : y \in X, y \neq x\}.$$

7. Proof

- Let us denote $f_-(x) \stackrel{\text{def}}{=} \inf\{f(y) : y \in X, y \neq x\}$ and $f_+(x) \stackrel{\text{def}}{=} \sup\{f(y) : y \in X, y \neq x\}$.
- Then, the formula for $F(x)$ takes the form

$$F(x) = f_{\&}\left(\frac{f(x) - \min(f(x), f_-(x))}{\max(f(x), f_+(x)) - \min(f(x), f_-(x))}, \mu(x)\right).$$

- If $\underline{f}(x) \leq \overline{f_-}(x)$, then for $f(x) = \underline{f}(x)$ and $f(y) = \overline{f}(y)$ for all $y \neq x$, we have $f(x) \leq f_-(x)$.
- Hence $\min(f(x), f_-(x)) = f(x)$.
- So, $f(x) - \min(f(x), f_-(x)) = 0$, and the ratio $F(x)$ takes its smallest possible value 0.
- If $\underline{f}(x) > \overline{f_-}(x)$, this implies that $f(x) > f_-(x)$ for all possible functions f, thus

$$F(x) = f_{\&}\left(\frac{f(x) - f_-(x)}{\max(f(x), f_+(x)) - f_-(x)}, \mu(x)\right).$$

8. Proof (cont-d)

- This expression is (non-strictly) increasing in $\mu(x)$ and decreasing in $f_+(x)$.
- Thus its minimum is attained when:
  - $\mu(x)$ attains its smallest possible value $\underline{\mu}(x)$ and
  - $f_+(x)$ attains its largest possible value $\overline{f_+}(x)$:

$$F(x) = f_\& \left(\frac{f(x) - f_-(x)}{\max(f(x), \overline{f_+}(x)) - f_-(x)}, \underline{\mu}(x) \right).$$

- If $\underline{f}(x) \geq \overline{f_+}(x)$, then $f(x) \geq \overline{f_+}(x)$, and thus, the ratio is always equal to its largest possible value 1.
- If $\underline{f}(x) < \overline{f_+}(x)$, then for $f(x) < \overline{f_+}(x)$, the ratio can be smaller than 1 and is equal to $\dfrac{f(x) - f_-(x)}{\overline{f}(x) - f_-(x)}$.

9. Proof (cont-d)

- This expression is increasing with $f(x)$.
- So its minimum is attained when $f(x)$ attains its smallest possible value $\underline{f}(x)$, thus

$$F(x) = f_{\&}\left(\frac{\underline{f}(x) - f_-(x)}{\max(\underline{f}(x), \overline{f_+}(x)) - f_-(x)}, \underline{\mu}(x)\right).$$

- To find the dependence on $f_-(x)$, we can represent the ratio as

$$\frac{\underline{f}(x) - f_-(x)}{\max(\underline{f}(x), \overline{f_+}(x)) - f_-(x)} = 1 - \frac{\max(\underline{f}(x), \overline{f_+}(x)) - \underline{f}(x)}{\max(\underline{f}(x), \overline{f_+}(x)) - f_-(x)}.$$

- This expression is clearly decreasing in $f_-(x)$.
- So its minimum is attained when $f_-(x)$ attains its largest possible value $f_-(x) = \overline{f_-}(x)$.
- Thus, we get the desired formula for $\underline{F}(x)$.

10. Proof (cont-d)

- Similar arguments explain the formula for $\overline{F}(x)$.

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