Interval (Set) Uncertainty as a Possible Way to Avoid Infinities in Physical Theories

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1. Infinities in Physical Theories: a Problem

- In many physical computations, we get meaningless infinite values for the desired quantities.
- This can be illustrated on an example of a simple problem:
  - computing the overall energy of the electric field
  - of a charged elementary particle.
- Due to relativity theory, an elementary (un-divisible) particle is a single point.
- Indeed, otherwise, the particle would:
  - contrary to the elementarity assumption,
  - consist of several not-perfectly-correlated parts.

2. Infinities in Physical Theories (cont-d)

- The energy E can be obtained by integrating the energy density $\rho(x)$ over the whole space:

$$E = \int \rho(x)\,dx.$$

- The energy density is proportional to the square of the electric field $E(x)$:

$$\rho(x) \sim E^2(x).$$

- Due to Coulomb Law, we have $E(x) \sim \dfrac{1}{r^2}$, where r is the distance to the particle's center.
- Thus, $\rho(x) = \dfrac{c}{r^4}$.

3. Infinities in Physical Theories (cont-d)

- Since this expression is spherically symmetric, we have $dx = 4\pi \cdot r^2\,dr$, hence

$$E = \int \rho(x)\,dx = \int_0^\infty \frac{c}{r^4} \cdot 4\pi \cdot r^2\,dr =$$

$$c \cdot \int_0^\infty r^{-2}\,dr = r^{-1}\Big|_{r=0}^{\infty} = \infty.$$

- This problem remains when we consider quantum equations instead of classical ones.

4. How This Problem Is Solved Now

- The usual way to solve this problem is by *renormalization*: crudely speaking,
  - we assume that the proper mass m_0 of the particle is $m_0 = -\infty$,
  - then the overall energy $m_0 \cdot c^2 + E$ is finite.
- To be more precise, we consider particles of radius ε, in which case the energy $E(\varepsilon)$ is large but finite.
- We take a mass $m_0(\varepsilon)$ for which $m_0(\varepsilon) \cdot c^2 + E(\varepsilon)$ is equal to the-determined finite value.
- Then, we tend ε to 0.

5. Problem with This Solution

- From the purely computational viewpoint, renormalization often works.
- However, it is not a physically meaningful procedure.
- To many physicists, it looks more like a mathematical trick.

6. Let Us Try to Find a More Physically Meaningful Solution to the Problem

- One of the most important physical features of quantum physics is its uncertainty.
- In the traditional non-quantum physics, equations are deterministic.
- In quantum physics:
  - we can only measure the state of the article with uncertainty, and
  - we can only predict the probabilities of different future events,
  - we can no longer predict the actual events with 100% guarantee.

7. Towards a More Physically Meaningful Solution (cont-d)

- It is therefore reasonable to use this feature to solve the above problem.
- We usually consider the particle located at one specific point.
- Instead, let us take into account that:
  - due to quantum-related uncertainty,
  - the particle can turn out to be at different locations.

8. A Usual Quantum-Physics Way of Describing Uncertainty Still Leads to Infinities

- Uncertainty in quantum physics is usually described by a probability distribution.
- Let us thus try to use this approach.
- For example, let us assume that the particle is
  - located within distance $\varepsilon > 0$ from the origin,
  - with, e.g., uniform distribution.
- This distribution corresponds to equal probability $\rho(z) =$ const of finding its location z anywhere within the corresponding sphere.
- Then, for each point x, the energy $E(x) = \dfrac{c}{|x - z|^4}$ depends on z.

9. Probabilistic Approach (cont-d)

- So it makes sense to consider the *average* energy

$$\int \frac{c}{|x-z|^4} \cdot \rho(z)\,dz.$$

- However, in the vicinity $z \approx x$, we have the same divergent integral as before.
- In a nutshell:
  - if we use probabilistic approach to describe the corresponding uncertainty,
  - the problem gets only worse.
- At least, before we had a finite value for the energy density.
- Now even energy density itself is infinite.

10. Probabilistic Approach (cont-d)

- We have shown that the value is infinite for the uniform distribution.
- However, one can show that it is infinite for any distribution with $\rho(z) > 0$ for some z.

11. It Is Thus Reasonable to Consider Interval (Set) Uncertainty

- We do not know the exact location of a particle.
- So it is not reasonable to assume that we know the exact probabilities either.
- The simplest case is:
  - when we have no information about the corresponding probabilities,
  - when all we know is, e.g., that the particle is located within the sphere of radius ε.
- Since we do not know probabilities, we cannot compute the average density.
- We can only compute, for each point x, the smallest and largest possible values of the energy density.

12. Interval (Set) Uncertainty (cont-d)

- For a point at distance r from the center:
  - the smallest possible value of the energy density is
  - when the particle is the farthest away from this point, at the distance $r+\varepsilon$.
- The largest possible value is when the particle is the closest, at distance $\max(r-\varepsilon, 0)$.
- For the largest density $\overline{\rho}(x) = \dfrac{c}{(\max(r-\varepsilon, 0))^4}$, we still get infinite overall energy.
- However, for the smallest energy density $\underline{\rho}(x) = \dfrac{c}{(r+\varepsilon)^4}$, we get a finite overall energy.
- And we get this finite value no matter what is the shape of the region in which the particle is located.

13. Our Hope

- This example makes us believe that:
  - such an interval (set) uncertainty
  - can help avoid infinities in other situations as well.

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