Kinematic Metric Spaces Under Interval Uncertainty: Towards an Adequate Definition

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1. What is a kinematic metric: physical introduction

- In the physical space, we can define the distance $d(a, b)$ between two points as the length of the shortest possible path between them.
- Thus defined distance is symmetric ($d(a, b) = d(b, a)$) and satisfies the usual triangle inequality $d(a, c) \le d(a, b) + d(b, c)$.
- The mathematical notion of a metric is a natural generalization of this physical notion.
- From the viewpoint of space-time:
  - physical space corresponds to the situation when
  - we take space-time points ("events") (a, t_0), (b, t_0), etc. corresponding to the same moment of time t_0.
- In relativity theory, such events cannot causally influence each other.
- Some events a *can* causally influence events b.
- We will denote this strict order – i.e., irreflexive transitive – relation by $a < b$.

2. What is a kinematic metric (cont-d)

- The causal influence is implemented by a particle or particles whose trajectories start at a and end up at b.
- For each such trajectory, we can measure the proper time of the corresponding particle.
- In principle, particles can travel as close to the speed of light as possible.
- In this case, the proper time can be as close to 0 as possible.
- So the *smallest* proper time over all trajectories is always 0.
- Interestingly, there is the *largest* proper time $\tau(a, b)$ – which corresponds to inertial motion.
- The corresponding function $\tau(a, b)$ – defined only when $a < b$ – satisfies the "anti-triangle" inequality $\tau(a, c) \geq \tau(a, b) + \tau(b, c)$.

3. What is a kinematic metric (cont-d)

- This inequality describes the known *twins paradox* of relativity theory:
  - when a twin brother who traveled to the stars comes back to Earth,
  - he will be younger than his twin who stayed on Earth.
- Indeed:
  - the biological age of the stay-home brother is $\tau(a, c)$, while
  - the biological age of the astronaut brother is $\tau(a, b) + \tau(b, c)$, where b is the moment when the brother reached a faraway star.
- A natural generalization of this function is a notion of *kinematic metric*.

4. Kinematic metric: definition

- Let $(X, <)$ be an ordered set.
- A function $\tau(a, b)$ – defined for all pairs for which $a < b$ – is called a *kinematic metric* if:
  - all its values are non–negative and
  - it satisfies the following "anti-triangle" inequality:

$$\tau(a, c) \geq \tau(a, b) + \tau(b, c).$$

5. Need for interval uncertainty

- All information about the values of a physical quantity v – including the values of the kinematic metric – comes from measurements.
- Measurements are never absolutely accurate.
- So the measurement result $\widetilde{v}$ is, in general, different from the actual (unknown) value v: there is a measurement error $\Delta v \stackrel{\text{def}}{=} \widetilde{v} - v$.
- Often, the only information that we have about the measurement error is an upper bound Δ on its absolute value.
- In this case, the only information that we have about the actual value v is that this value belongs to the interval

$$[\underline{v}, \overline{v}] \stackrel{\text{def}}{=} [\widetilde{v} - \Delta, \widetilde{v} + \Delta].$$

6. Natural question

- Suppose that we have, for all pairs $a < b$, intervals $[\underline{\tau}(a,b), \overline{\tau}(a,b)]$, with $\underline{\tau}(a,b) \geq 0$, obtained from measurement.
- Here, $[\underline{\tau}(a,b), \overline{\tau}(a,b)] = [\widetilde{\tau}(a,b) - \Delta(a,b), \widetilde{\tau}(a,b) + \Delta(a,b)]$.
- If all the upper bounds $\Delta(a,b)$ are correct, then there is a kinematic metric $\tau(a,b)$ for which $\tau(a,b) \in [\underline{\tau}(a,b), \overline{\tau}(a,b)]$ for all $a < b$.
- However, if we – as happens – underestimated the measurement errors, we may not have such a function.
- So, a natural question is: what is the condition on the intervals $[\underline{\tau}(a,b), \overline{\tau}(a,b)]$ under which such a function $\tau(a,b)$ exists?

7. A seemingly natural idea does not work

- Anti-triangle inequality implies that

$$\overline{\tau}(a,c) \geq \underline{\tau}(a,b) + \underline{\tau}(b,c) \text{ for all } a < b < c.$$

- So, it may seem that this inequality is the right condition for the existence of the desired kinematic metric $\tau(a,b)$.
- However, this inequality does not guarantee the existence of $\tau(a,b)$.
- For example, for $X = \{a_1 < a_2 < a_3 < a_4\}$ and $[\underline{\tau}(a_i,a_j), \overline{\tau}(a_i,a_j)] = [1,2]$ for all $i < j$:
  - this inequality is satisfied, but
  - the desired function $\tau(a,b)$ is not possible.
- Indeed, if $\tau(a,b)$ existed, we would have:

$$2 \geq \tau(a_1,a_4) \geq \tau(a_1,a_2) + \tau(a_2,a_3) + \tau(a_3,a_4) \geq 3, \text{ i.e., } 2 \geq 3.$$

8. Main result

- For an interval-valued function $[\underline{\tau}(a,b), \overline{\tau}(a,b)]$ defined for all $a < b$, the following two conditions are equivalent to teach other:
  - there exists a kinematic metric $\tau(a,b)$ for which always
  $$\tau(a,b) \in [\underline{\tau}(a,b), \overline{\tau}(a,b)];$$
  - we have $\overline{\tau}(a_1, a_n) \geq \sum\limits_{i=1}^{n-1} \underline{\tau}(a_i, a_{i+1})$ for all sequences
  $$a_1 < \ldots < a_n.$$

9. Proof

- If $\tau(a, b)$ exists, then this inequality is clearly satisfied.
- Indeed, it follows from the anti-triangle inequality

$$\overline{\tau}(a_1, a_n) \geq \tau(a_1, a_n) \geq \sum_{i=1}^{n-1} \tau(a_i, a_{i+1}) \geq \sum_{i=1}^{n-1} \underline{\tau}(a_i, a_{i+1}).$$

- Vice versa, suppose that the above condition is satisfied.
- Then, we can take $\tau(a, b) = \sup \left\{ \sum_{i=1}^{n-1} \underline{\tau}(a_i, a_{i+1}) \right\}$.
- Here, the supremum is taken over all the chains

$$a = a_1 < a_2 < \ldots < a_n = b \text{ that connect } a \text{ and } b.$$

- One can easily prove that thus defined function satisfies the anti-triangle inequality.
- By taking a chain $a_1 = a < a_2 = b$, we get $\tau(a, b) \geq \underline{\tau}(a, b)$.

10. Proof (cont-d)

- From the above inequality, for each chain, we get

$$\overline{\tau}(a,b) \geq \sum_{i=1}^{n-1} \underline{\tau}(a_i, a_{i+1}).$$

- Since $\overline{\tau}(a,b)$ is greater than or equal to each sum, it is greater than or equal to their supremum:

$$\overline{\tau}(a,b) \geq \sup \left\{ \sum_{i=1}^{n-1} \underline{\tau}(a_i, a_{i+1}) \right\}.$$

- Thus, $\underline{\tau}(a,b) \leq \tau(a,b) \leq \overline{\tau}(a,b)$, i.e.:

$$\tau(a,b) \in [\underline{\tau}(a,b), \overline{\tau}(a,b)].$$

11. Comment

- We need the above condition for *all* natural numbers n:

$$\overline{\tau}(a_1, a_n) \geq \sum_{i=1}^{n-1} \underline{\tau}(a_i, a_{i+1}) \text{ for all sequences } a_1 < \ldots < a_n.$$

- If we only require it only for $n \leq n_0$, this does not guarantee the existence of $\tau(a, b)$.
- Example:
  - $X = \{a_1 < \ldots < a_{n_0+1}\}$ and
  - $[\underline{\tau}(a_i, a_j), \overline{\tau}(a_i, a_j)] = [1, n_0 - 1]$ for all $i < j$.
- Indeed, if $\tau(a, b)$ existed, we would have

$$n_0 - 1 \geq \tau(a_1, a_{n_0+1}) \geq \tau(a_1, a_2) + \ldots + \tau(a_{n_0}, a_{n_0+1}) \geq n_0,$$

$$\text{i.e., } n_0 - 1 \geq n_0.$$

12. Acknowledgments

This work was supported in part by the National Science Foundation grants:

- 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science);
- HRD-1834620 and HRD-2034030 (CAHSI Includes).

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478.

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