Inconsistencies in Fuzzy Estimations: Kaucher Arithmetic Naturally Appears

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1. Fuzzy techniques: a brief reminder

- When experts describes how they solve tasks, they usually use imprecise ("fuzzy") words from natural language such as *small*.
- This is, e.g., how expert drivers control their cars.
- To describe this knowledge in precise computer-understandable terms, Lotfi Zadeh proposed to ask the expert to assign:
  - to each possible value x of the corresponding property,
  - a degree $m \in [0,1]$ to which x satisfies the property – e.g., to which x is small.
- He called this technique *fuzzy*.

2. Fuzzy logic: a brief reminder

- Experts often use logical connectives $*$ – e.g., “and” and “or” – in describing their decisions.
- For example, a condition for a certain action may be that a car in front is close *and* that it brakes a little bit.
- To estimate the degree c of a statement $A * B$ based on degrees a and b of its component statements A and B, Zadeh proposed to take

$$c = f(a, b).$$

- Here, $f : [0, 1] \times [0, 1] \to [0, 1]$ is a (non-strictly) increasing continuous function for which:
  - for $a, b \in \{0, 1\}$,
  - the value $f(a, b)$ is the usual truth value of the corresponding logical operation.

3. Fuzzy logic: successes and challenges

- Fuzzy approach led to many successes.
- However, its representation of expert knowledge was not always perfectly adequate.
- Two ideas were proposed to make it more adequate.

4. Interval-valued fuzzy techniques

- One of the ideas was to take into account that:
  - just like an expert cannot describe the exact value of control,
  - e.g., he/she only says "a little bit",
  - this same expert cannot meaningfully describe his/her degree of belief by a single number.
- The expert's opinion would be described more adequately if we allow the expert to use the interval $[\underline{m}, \overline{m}]$ of possible values.
- This is known as *interval-valued* fuzzy logic.

5. Interval-valued fuzzy techniques (cont-d)

- In line with general interval techniques:
  - once we know the degrees $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ of statements A and B,
  - it is reasonable to estimate the degree of $A * B$ as

$$f([\underline{a}, \overline{a}], [\underline{b}, \overline{b}]) \stackrel{\text{def}}{=} \{f(a, b) : a \in [\underline{a}, \overline{a}], b \in [\underline{b}, \overline{b}]\}.$$

- Since $f(a, b)$ is increasing, this leads to

$$f([a], [b]) = [f(\underline{a}, \underline{b}), f(\overline{a}, \overline{b})].$$

6. Second idea: intuitionistic fuzzy techniques

- The second idea was to take into account that:
  - when the expert is not 100% sure that x is small,
  - this means that he/she also has arguments that x is not small.
- So, in addition to the degree m to which the property is true, it makes sense to also ask for the degree m_- to which this property is false.
- This is known as *intuitionistic* fuzzy logic.
- We assume that there is no inconsistency, so $m + m_- \le 1$.
- Then, when we have pairs (a, a_-) and (b, b_-) corresponding to A and B, it is reasonable to get $(c.c_-)$, where:
  - we apply f to a and b: $c = f(a, b)$, and
  - we apply a dual operation to a_- and b_-: $c_- = 1 - f(1 - a_-, 1 - b_-)$.

7. From the purely mathematical viewpoint, these ideas are equivalent

- These two ideas have a different meaning.
- However, from the purely mathematical viewpoint, they are equivalent:
  - we can map an interval $[\underline{x}, \overline{x}]$ to a pair $(\underline{x}, 1 - \overline{x})$ and,
  - vice versa, we can map a pair (a, a_-) to an interval $[a, 1 - a_-]$.
- Then, all operations remain the same.
- Namely, we get the result of an intuitionistic fuzzy operation if we:
  - transform each intuionistic value into an interval,
  - apply the interval operation, and
  - transform the resulting interval back into intuitionistic fuzzy value.

8. Possible inconsistencies naturally lead to improper intervals

- In some cases, there is an inconsistency between arguments for and against the same statement.
- In such cases, it makes sense to consider pairs (a, a_-) for which

$$a + a_- > 1.$$

- We can still apply operation $f(a, b)$ to such values:

$$f((a, a_-), (b, b_-)) \stackrel{\text{def}}{=} (f(a, b), 1 - f(1 - a_-, 1 - b_-)).$$

- If we apply the above-mentioned transformation $(a, a_-) \mapsto [a, 1 - a_-]$ to such pairs, we get an *improper* interval $[\overline{a}, \underline{a}]$ for which $\overline{a} > \underline{a}$.

9. Possible inconsistencies naturally lead to Kaucher arithmetic

- There is a special arithmetic – first proposed by Kaucher – extending interval arithmetic to improper intervals.
- The usual interval operations describe the set of all possible values $f(a, b)$ when $a \in [\underline{a}, \overline{a}]$ and $b \in [\underline{b}, \overline{b}]$.
- For an increasing function f, we have

$$f([\underline{a}, \overline{a}], [\underline{b}, \overline{b}]) = [f(\underline{a}, \underline{b}), f(\overline{a}, \overline{b}]).$$

- When both inputs are improper, we get a similar formula

$$f([\overline{a}, \underline{a}], [\overline{b}, \underline{b}]) = [f(\overline{a}, \overline{b}), f(\underline{a}, \underline{b})].$$

10. Possible inconsistencies naturally lead to Kaucher arithmetic (cont-d)

- When one input is proper $[\underline{a}, \overline{a}]$ and one improper $[\overline{b}, \underline{b}]$, the result can be defined as $[\underline{c}, \overline{c}]$ or $[\overline{c}, \underline{c}]$, where:
  - the interval $[\underline{c}, \overline{c}]$ is the intersection of all possible ranges of $f(S)$
  - over all connected sets $S \subseteq [\underline{a}, \overline{a}] \times [\underline{b}, \overline{b}]$ whose projections to a- and b-axis are exactly $[\underline{a}, \overline{a}]$ and $[\underline{a}, \overline{a}]$.
- For continuous monotonic functions f, this implies that

$$f([\underline{a}, \overline{a}], [\overline{b}, \underline{b}]) = [f(\underline{a}, \overline{b}), f(\overline{a}, \underline{b})].$$

- Our main observation is that we get exactly $f((a, a_-), (b, b_-))$ if we:
  - first transform intuitionistic fuzzy values (a, a_-) and (b, b_-) into intervals (proper or improper),
  - apply Kaucher operation to these intervals, and
  - transform the results back into an intuitionistic fuzzy value.

11. Comment: other relations between fuzzy and Kaucher arithmetic

- Suppose that we know the sets $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ of all possible values of a and b.
- Then the set of all *potentially possible* values of $a + b$ is the interval sum $[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}]$.
- Here, the Kaucher sum $[\underline{a}, \overline{a}] + [\overline{b}, \underline{b}]$ is the set of all *definitely possible* values.
- These two intervals are the simplest case of a family of embedded intervals – which is exactly what a fuzzy number is.

12. References

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