Shapley Value Under Interval Uncertainty Revisited: Why Seemingly Natural Axiomatic Approach Is Not Fully Adequate

Miroslav Svitek[1], Olga Kosheleva[2] and Vladik Kreinovich[3]

[1] Faculty of Transportation Sciences, Czech Technical University in Prague
110 00 Prague 1, Czech Republic, miroslav.svitek@cvut.cz
[2,3]Departments of [2]Teacher Education and [3]Computer Science
University of Texas at El Paso, 500 W. University
El Paso, Texas 79968, USA
olgak@utep.edu, vladik@utep.edu

1. Shapley value: a brief reminder

- Many successes are due to collaboration, be it in manufacturing or in research.
- How to fairly divide the dividends between all n participants?
- For example, when we evaluate individual researchers, how to fairly distribute the overall points-for-paper between paper co-authors?
- In this division, it is reasonable to take into account:
  - for each set $S \subseteq N \stackrel{\text{def}}{=} \{1, \ldots, n\}$,
  - what would be the productivity $v(S)$ if only participants from the set S worked together.
- The answer to this question was produced by the future Nobelist Lloyd Shapley.

2. Shapley value: a brief reminder (cont-d)

- Shapley formulated natural conditions:
  - additivity;
  - symmetry; and
  - that a person who does not contribute anything, i.e., for whom $v(S\cup\{i\})=v(S)$ for all S, should not get anything.
- He proved that there is only one distribution scheme that satisfies these conditions, in which Person i gets the amount

$$x_i(v)=\sum a(|S|)\cdot(v(S\cup\{i\})-v(S)).$$

- Here, the sum is taken overall all sets S for which $i\not\in S$, $|S|$ denoted the number of elements in a set S, and

$$a(m)\stackrel{\text{def}}{=}\frac{m!\cdot(n-m)!}{n!}.$$

- This expression for $x_i(v)$ is known as the *Shapley value.*

3. Shapley value: a brief reminder (cont-d)

- Lately, Shapley value has also been actively used in machine learning.
- There, it is used to decide which of n features used to make a decision are most important.
- In this case, $v(S)$ is the effectiveness that we get when we only use features from the set S.

4. Need for interval uncertainty

- In practice, we rarely know the exact values $v(S)$.
- Often, we only know an interval $[v](S) = [\underline{v}(S), \overline{v}(S)]$ that contains $v(S)$.
- The agreement about division is usually decided before the project starts, in which case even the future value $v(N)$ is not known exactly.
- In this case, a reasonable idea is to come up with intervals

$$[x]_i([v]) = [\underline{x}_i([v]), \overline{x}_i([v])].$$

- Then we can use Hurwicz approach and make a distribution

$$x_i(v) = \alpha \cdot \overline{x}_i(v) + (1 - \alpha) \cdot \underline{x}_i(v).$$

- Here α is determined from the condition that:
  - the sum of these values should be equal to
  - the overall amount $v(N)$ – the overall monetary amount or the overall number of points for this particular paper.

5. Current interval method and its limitation

- A recent paper considers similar conditions to Shapley's.
- It shows that:
  - under these conditions,
  - we should take, as bounds on $x_i([v])$, the Shapley values corresponding to the functions $\underline{v}(S)$ and $\overline{v}(S)$.
- In many cases, this approach leads to reasonable results, but in other cases, it does not.
- For example, for $n = 2$:
  - if $\underline{v}(S) = 0$ for all S, $\overline{v}(\emptyset) = \overline{v}(\emptyset) = \overline{v}(\{2\}) = 0$, and $\overline{v}(\{1\}) = \overline{v}(\{1, 2\}) = 1$,
  - then Person 2 gets nothing.

6. Current interval method and its limitation (cont-d)

- However, it is possible, e.g., that the actual values are $v(\{1,2\}) = 1$ and $v(\{1\}) = v(\{2\}) = 0$.
- In this case, due to symmetry, Person 2 should get exactly the same amount as Person 1.

7. Analysis of the problem and resulting solution

- The reason for the above problem is as follows/
- For exact values $v(S)$, the condition that $v(S \cup \{i\}) = v(S)$ for all S indeed means that i did not contribute anything,
- However, as the above example shows:
  - a similar interval equality $[v](S \cup \{i\}) = [v](S)$ for all S
  - does not necessarily imply that Person i was not contributing.
- So, a natural idea is to take, as $[x]_i([v])$, the set of all possible values $x_i(v)$ for all functions v for which $v(S) \in [v](S)$ for all S.

8. Resulting solution (cont-d)

- To find these intervals, let us take into account that the Shapley value formula can be reformulated as

$$x_i(v) = \sum_{S:i\in S} a(|S|+1)\cdot v(S) - \sum_{S:i\notin S} a(|S|)\cdot v(S).$$

- Thus, by using usual interval computations, we get:

$$\underline{x}_i([v]) = \sum_{S:i\in S} a(|S|+1)\cdot \underline{v}(S) - \sum_{S:i\notin S} a(|S|)\cdot \overline{v}(S);$$

$$\overline{x}_i([v]) = \sum_{S:i\in S} a(|S|+1)\cdot \overline{v}(S) - \sum_{S:i\notin S} a(|S|)\cdot \underline{v}(S);$$

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