Interval Computations as Applied Constructive Mathematics: from Shanin to Wiener and Beyond

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1. General Problems of Science and Engineering

- The main objective of science is to understand the current state of the world and to predict its future state.
- The main objective of engineering is to find controls and strategies that lead to a better future.
- The state of the world is usually described in terms of real numbers – values of physical quantities.
- Some quantities we can measure directly: e.g., distance from here to our hotel.
- Other quantities y we cannot measure directly: e.g., distance from here to a nearby star.

2. Indirect Measurements

- Since we cannot measure the quantity of interest y directly, we measure it *indirectly*.
- Namely, we measure related easier-to-measure quantities $x_1, \ldots, x_n$ and get values $\widetilde{x}_i$.
- Then, we use the known relation $y = f(x_1, \ldots, x_n)$ and known (approximate) values of x_i to estimate y as

$$\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n).$$

3. How Constructive Mathematics Can Help

- Ideally, we want to be able to estimate y with any given accuracy ε.
- For this purpose, we need to have:
  - an algorithm that, given ε, computes the accuracy δ with we should measure the inputs, and
  - an algorithm $\widetilde{f}$ that, when applied to the measurement results $\widetilde{x}_i$ to get the desired estimate $\widetilde{y}$:

$$|\widetilde{x}_i - x_i| \le \delta \Rightarrow |\widetilde{f}(\widetilde{x}_1, \ldots, \widetilde{x}_n) - f(x_1, \ldots, x_n)| \le \varepsilon.$$

- In a nutshell, this is what constructive mathematics is about – when limited to real numbers.

4. Applied Constructive Mathematics

- In practice, our ability to measure accurately is limited.
- So, we have measurement results $\widetilde{x}_i$ with some accuracies δ_i: $|\widetilde{x}_i - x_i| \le \delta_i$.
- The only information that we have about the actual value x_i is that $x_i \in [\underline{x}_i, \overline{x}_i] \stackrel{\text{def}}{=} [\widetilde{x}_i - \delta_i, \widetilde{x}_i + \delta_i]$.
- What can we say about $y = f(x_1, \ldots, x_n)$? We can only conclude that

$$y \in [\underline{y}, \overline{y}] \stackrel{\text{def}}{=} \{f(x_1, \ldots, x_n) : x_i \in [\underline{x}_i, \overline{x}_i]\}.$$

- Computing $[\underline{y}, \overline{y}]$ is called *interval computations.*
- Yuri Matiyasevich called it *applied constructive mathematics.*

5. Why Intervals?

- Usually, we do not just know the upper bound δ_i on the measurement error $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$: $|\Delta x_i| \le \delta_i$.
- We also know the probabilities of different values Δx_i.
- These probabilities come from comparing measurement results with a standard (more accurate) instrument.
- There are two situations when this is not possible:
  - state-of-the-art measurement, when we use the most accurate instrument; and
  - measurements on the shop floor, where we could calibrate everything, but it would cost too much.
- Then, all we have is an upper bound δ_i on $|\Delta x_i|$.

6. Why Wiener? A Brief History of Interval Computations

- *Origins:* Archimedes (Ancient Greece), N. Wiener (1914)
- *Modern pioneers:* Mieczyslaw Warmus (Poland), Teruo Sunaga (Japan), Ramon Moore (USA), 1956–59
- *First boom:* early 1960s.
- *First challenge:* taking interval uncertainty into account when planning spaceflights to the Moon.
- *Current applications* (sample):
  - design of elementary particle colliders: Martin Berz, Kyoko Makino (USA)
  - will a comet hit the Earth: Martin Berz, Ramon Moore (USA)
  - robotics: L. Jaulin (France), A. Neumaier (Austria)
  - chemical engineering: M. Stadtherr (USA)

7. Interval Computations – How? First Idea

- In a computer, every computation is a sequence of elementary arithmetic operations.
- In mathematical terms, this means that we consider compositions of simple arithmetic functions.
- So, a natural idea – known as *straightforward interval computations* – is to:
  - find interval analogues of simple arithmetic functions, and then
  - in the original algorithm, replace each arithmetic operation with the corresponding interval one.

8. Interval Analogues of Simple Arithmetic Functions

- When $x_1 \in \mathbf{x}_1 = [\underline{x}_1, \overline{x}_1]$ and $x_2 \in \mathbf{x}_2 = [\underline{x}_2, \overline{x}_2]$, then:
  - The range $\mathbf{x}_1 + \mathbf{x}_2$ for $x_1 + x_2$ is $[\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2]$.
  - The range $\mathbf{x}_1 - \mathbf{x}_2$ for $x_1 - x_2$ is $[\underline{x}_1 - \overline{x}_2, \overline{x}_1 - \underline{x}_2]$.
  - The range $\mathbf{x}_1 \cdot \mathbf{x}_2$ for $x_1 \cdot x_2$ is $[\underline{y}, \overline{y}]$, where

$$\underline{y} = \min(\underline{x}_1 \cdot \underline{x}_2, \underline{x}_1 \cdot \overline{x}_2, \overline{x}_1 \cdot \underline{x}_2, \overline{x}_1 \cdot \overline{x}_2);$$

$$\overline{y} = \max(\underline{x}_1 \cdot \underline{x}_2, \underline{x}_1 \cdot \overline{x}_2, \overline{x}_1 \cdot \underline{x}_2, \overline{x}_1 \cdot \overline{x}_2).$$

- The range $1/\mathbf{x}_1$ for $1/x_1$ is $[1/\overline{x}_1, 1/\underline{x}_1]$ (if $0 \notin \mathbf{x}_1$).
- These operations are known as *interval arithmetic*.

9. Straightforward Interval Computations: Example and Limitations

- *Example:* $f(x) = (x-2)\cdot(x+2)$, $x \in [1,2]$.
- How will the computer compute it?
  - $r_1 := x - 2$;
  - $r_2 := x + 2$;
  - $r_3 := r_1 \cdot r_2$.
- *Main idea:* perform the same operations, but with *intervals* instead of *numbers*:
  - $\mathbf{r}_1 := [1,2] - [2,2] = [-1,0]$;
  - $\mathbf{r}_2 := [1,2] + [2,2] = [3,4]$;
  - $\mathbf{r}_3 := [-1,0] \cdot [3,4] = [-4,0]$.
- *Actual range:* $f(\mathbf{x}) = [-3,0] \subset [-4,0]$.
- *Comment:* excess width (4 vs. 3) is unavoidable, since interval computations is NP-hard.

10. What Can We Do?

- Representing an algorithm as a composition of elementary arithmetic functions often does not work.
- Idea: represent it as a composition of some other functions.
- What is the class of functions closed under composition?
- It is reasonable to require that this class is also closed under inversion.
- So, we are looking for *group* of transformations of $\mathbb{R}^n$.
- The simplest such group is the group of all linear transformations.
- What are other such groups?

11. Wiener Again

- N. Wiener is mostly known as the father of *cybernetics*, a general theory of biological and engineering systems.
- His interest started when a physiologist noticed that his design resembled the actual neural structure.
- Wiener noticed that when approach a faraway object, we go through five phases.
- At first, we notice a blur – corresponding to all possible transformations.
- Then, we get a shape modulo projective transformations.
- Then, affine, then homotheties, and finally, we identify the object exactly.

12. Wiener (cont-d)

- Wiener mentioned that we are a product of billion years of improving evolution.
- So, if there were other groups, we would have used them.
- So, he conjectured that there are no other groups.
- The only transformation groups containing all linear one are all projective ones and all transformation.
- Surprisingly, this was indeed proven in the 1960s by V. M. Guillemin, I. M. Singer, and S. Sternberg.

13. For the Resulting Transformations, Interval Computations Are Feasible

- The most general case is all possible transformations.
- Locally – in the vicinity of id – they are monotonic.
- Computing the range of monotonic $f(x_1, \ldots, x_n)$ is easy.
- For example, if f increases in all x_i, the range is

$$[f(\underline{x}_1, \ldots, \underline{x}_n), f(\overline{x}_1, \ldots, \overline{x}_n)].$$

- The range of a linear $f(x_1, \ldots, x_n) = a_0 + \sum\limits_{i=1}^{n} a_i \cdot x_i$ on $[\widetilde{x}_i - \delta_i, \widetilde{x}_i + \delta_i]$ is $[\widetilde{y} - \delta, \widetilde{y} + \delta]$, where:

$$\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n) \text{ and } \delta = \sum_{i=1}^{n} |a_i| \cdot \delta_i.$$

- Feasible algorithms are known for fractional-linear f.

14. Towards Standard Interval Computations

- If a function is linear or fractional-linear, we apply the known algorithms.
- If not, we check whether the function is monotonic.
- We know that $f \uparrow x_i$ if $\dfrac{\partial f}{\partial x_i} \geq 0$.
- So, to check, we estimate the range $[\underline{d}_i, \overline{d}_i]$ of this partial derivative – e.g., by straightforward interval comp.
- If $\underline{d}_i \geq 0$, we can use monotonicity-based formulas.
- If the function is not monotonic, we try to approximate it by one of the feasible-for-intervals functions.
- Approximations by monotonic functions is a new idea, currently being tested.
- Approximation by fractional-linear functions is a raw idea, no algorithm is know.

15. Standard Interval Computations (cont-d)

- Approximation by linear functions – 1st terms in Taylor series – is well known: for some $\eta \in [\underline{x}_1, \overline{x}_1] \times \ldots$:

$$f(x_1, \ldots, x_n) = f(\widetilde{x}_1, \ldots, \widetilde{x}_n) + \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x_i} \right|_{\eta} \cdot \Delta x_i.$$

- Thus, we get a *centered form* estimate

$$[\underline{y}, \overline{y}] \subseteq \widetilde{y} + \sum_{i=1}^{n} [\underline{d}_i, \overline{d}_i] \cdot [-\delta_i, \delta_i].$$

- This formula is obtained by ignoring second order terms, so its accuracy is $O(\delta_i^2)$.
- To increase its accuracy, we can decrease δ_i by bisection.

16. Bisection: How?

- If we bisect all the intervals, we get 2^n subboxes – too many for large n.
- So, we need to decided which interval to bisect, based on the values $|d_i|$ and δ_i.
- The resulting criterion $f(|d_i|, \delta_i)$ should not change if we change the units for measuring x_i or y:

$$f(|d_i|, \delta_i) > f(|c_i|, \gamma_i) \Leftrightarrow f(\mu \cdot \lambda^{-1} \cdot |d_i|, \lambda \cdot \delta_i) > f(\mu \cdot \lambda^{-1} \cdot |c_i|, \lambda \cdot \gamma_i).$$

- This implies bisecting where $|d_i| \cdot \delta_i \to \max$.

17. Resulting Algorithm

- We need to estimate the range of $f(x_1, \ldots, x_n)$ on intervals $[\underline{x}_i, \overline{x}_i] = [\widetilde{x}_i - \delta_i, \widetilde{x}_i + \delta_i]$.
- First, we use straightforward interval computations to find the range $[\underline{d}_i, \overline{d}_i]$ of each partial derivative $\dfrac{\partial f}{\partial x_i}$.
- If $\underline{d}_i \geq 0$ or $\overline{d}_i \leq 0$, monotonicity reduces the problem to a problem with $n-1$ variables.
- If the result is non-monotonic, we use the centered form estimate: $[\underline{y}, \overline{y}] \subseteq \widetilde{y} + \sum\limits_{i=1}^{n} [\underline{d}_i, \overline{d}_i] \cdot [-\delta_i, \delta_i]$.
- To get a more accurate estimate, we bisect the interval with the largest product $|d_i| \cdot \delta_i$, and repeat.

18. Monotonicity: Example

- *Idea:* if the range $[\underline{r}_i, \overline{r}_i]$ of each $\dfrac{\partial f}{\partial x_i}$ on $\mathbf{x}_i$ has $\underline{r}_i \geq 0$, then

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_n) = [f(\underline{x}_1, \ldots, \underline{x}_n), f(\overline{x}_1, \ldots, \overline{x}_n)].$$

- *Example:* $f(x) = (x-2)\cdot(x+2)$, $\mathbf{x} = [1, 2]$.
- *Case $n = 1$:* if the range $[\underline{r}, \overline{r}]$ of $\dfrac{df}{dx}$ on $\mathbf{x}$ has $\underline{r} \geq 0$, then

$$f(\mathbf{x}) = [f(\underline{x}), f(\overline{x})].$$

- *AD:* $\dfrac{df}{dx} = 1\cdot(x+2) + (x-2)\cdot 1 = 2x.$
- *Checking:* $[\underline{r}, \overline{r}] = [2, 4]$, with $2 \geq 0$.
- *Result:* $f([1,2]) = [f(1), f(2)] = [-3, 0]$.
- *Comparison:* this is the exact range.

19. Centered Form: Example

- *General formula:*

$$\mathbf{Y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \cdot [-\Delta_i, \Delta_i].$$

- *Example:* $f(x) = x \cdot (1 - x)$, $\mathbf{x} = [0, 1]$.
- Here, $\mathbf{x} = [\widetilde{x} - \Delta, \widetilde{x} + \Delta]$, with $\widetilde{x} = 0.5$ and $\Delta = 0.5$.
- *Case* $n = 1$*:* $\mathbf{Y} = f(\widetilde{x}) + \dfrac{df}{dx}(\mathbf{x}) \cdot [-\Delta, \Delta]$.
- *AD:* $\dfrac{df}{dx} = 1 \cdot (1 - x) + x \cdot (-1) = 1 - 2x$.
- *Estimation:* we have $\dfrac{df}{dx}(\mathbf{x}) = 1 - 2 \cdot [0, 1] = [-1, 1]$.
- *Result:* $\mathbf{Y} = 0.5 \cdot (1 - 0.5) + [-1, 1] \cdot [-0.5, 0.5] = 0.25 + [-0.5, 0.5] = [-0.25, 0.75]$.
- *Comparison:* actual range $[0, 0.25]$, straightforward $[0, 1]$.

20. Centered Form and Bisection: Example

- *Known:* accuracy $O(\Delta_i^2)$ of first order formula

$$f(x_1,\ldots,x_n)=f(\widetilde{x}_1,\ldots,\widetilde{x}_n)+\sum_{i=1}^n \frac{\partial f}{\partial x_i}(\chi)\cdot(x_i-\widetilde{x}_i).$$

- *Idea:* if the intervals are too wide, we:
  - split one of them in half ($\Delta_i^2\to\Delta_i^2/4$); and
  - take the union of the resulting ranges.
- *Example:* $f(x)=x\cdot(1-x)$, where $x\in\mathbf{x}=[0,1]$.
- *Split:* take $\mathbf{x}'=[0,0.5]$ and $\mathbf{x}''=[0.5,1]$.
- *1st range:* $1-2\cdot\mathbf{x}=1-2\cdot[0,0.5]=[0,1]$, so $f\uparrow$ and $f(\mathbf{x}')=[f(0),f(0.5)]=[0,0.25]$.
- *2nd range:* $1-2\cdot\mathbf{x}=1-2\cdot[0.5,1]=[-1,0]$, so $f\downarrow$ and $f(\mathbf{x}'')=[f(1),f(0.5)]=[0,0.25]$.
- *Result:* $f(\mathbf{x}')\cup f(\mathbf{x}'')=[0,0.25]$ – exact.

21. Wiener and Constructive Mathematics Yet Again

- As we have mentioned, the general problem of interval computations is NP-hard.
- It is NP-hard even for computing the range of sample variance $\frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - a)^2$, where $a = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$.
- So (unless P = NP), the *worst-case* complexity of interval computations problems is exponential.
- A natural question: what about *average* computational complexity?
- Here, we need a probability measure on the set of all functions.
- In this problem, Norbert Wiener was also a pioneer with his Wiener measure.

22. Wiener Yet Again (cont-d)

- The original formulas of Wiener were not algorithmic.
- In my early papers, it was shown that constructivization is possible.
- This was inspired by Shanin and constructive mathematics.
- That result was for Wiener's measure – and real distribution may be different.
- So, recently, we extended these algorithmic results to general probability measures over metric spaces.
- This includes function spaces as particular cases.

23. Beyond Intervals

- Instead of boxes, we can have other sets describing uncertainty – e.g., ellipsoids or zonotopes.
- Instead of numbers, we can have similar uncertainty about more complex objects – e.g., functions.
- This becomes applications of beyond-numbers constructive mathematics.
- All these problems can be naturally reformulated in terms of modal logic.
- Indeed, $x_i \in [\underline{x}_i, \overline{x}_i]$ means that all values from this interval are possible.
- We want to find when y is a possible value of $f(x_1, \ldots, x_n)$.
- Thus, it is also applied modal logic.
- Can Yu. Gurevich's modal constructive logic help?

24. Interval Computations: Everyone is Welcome

- Interval computations is extremely important for practice.
- There are many theoretical and practical open problems.
- We have regular biannual conferences SCAN'XX, the next one will be in Hungary in September 2020.
- There are annual European SWIM workshops.
- We have a journal *Reliable Computing* (formerly *Interval Computations*), founded by Yu. Matiyasevich.
- Our website is `http://www.cs.utep.edu/interval-comp`
- Everyone is welcome to visit – or even to join our community!