How the Amount of Cracks and Potholes Grows with Time: Symmetry-Based Explanation of Empirical Dependencies

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1. Cracks and Potholes

- When a road is built, it is almost perfect – it has only miniature cracks and potholes.
- However, as the road is used, cracks and potholes appear and start growing.
- The amount of cracks is gauged the overall length C of longitudal cracks outside the wheel path.
- The amount of potholes is usually gauged by the total area P of potholes.
- As the road is used, the quality of the pavement deteriorates, and the values C and P grow.
- This growth starts at some small values corresponding to the newly built road – age $t = 0$.

2. Cracks and Potholes (cont-d)

- It continues growing until they reach the maximum – the undesirable bad state.
- In this state, the whole road is covered by cracks and potholes.
- The empirical formulas for this growth are:

$$C = a_C\cdot\exp(-b_C\cdot\exp(-c_C\cdot t));\;\; P = a_P\cdot\exp(-b_P\cdot\exp(-c_P\cdot t)).$$

- In this talk, we use natural symmetry ideas to provide a theoretical explanation for these empirical formulas.

3. Natural Transformations

- In science and engineering, we are interested in the values of different physical quantities.
- We describe these quantities in numerical form.
- However, the numerical values of the corresponding quantities depend on the measuring unit.
- For some quantities such as temperature or time, the values also depend on the starting point.
- If we change the measuring unit for length from meters to centimeters, then all numerical values are $\times$ by 100.
- For example, 2 m becomes $2 \cdot 100 = 200$ cm.

4. Natural Transformations (cont-d)

- In general:
  - if we replace the original measuring unit with a new unit which is λ times smaller,
  - all numerical values are multiplied by λ:

$$x \to X = \lambda \cdot x.$$

- This numerical transformation is known as *scaling*.
- Similarly, we can start measuring time:
  - not from our year 0,
  - but – as the French Revolution suggested – with the year 1789 when the revolution started.
- Then from all year values, we should subtract 1789.

5. Natural Transformations (cont-d)

- In general:
  - if we replace the original starting point with the one which is x_0 units before,
  - then we add x_0 to all numerical values:

$$x \to X = x + x_0.$$

- This numerical transformation is known as *shift*.

6. Natural Symmetries

- For most physical quantities, there is no fixed measuring unit – and sometimes no fixed starting point.
- It is therefore reasonable to require that:
  - the dependencies $y = f(x)$ between physical quantities
  - also not depend on the choice of the measuring unit
  - (and possibly on the choice of the starting point).
- In physics, such invariance is called *symmetry*.

7. Natural Symmetries (cont-d)

- Of course:
  - if we just change the unit and/or starting point for x,
  - to keep the same formula true in the new units, we may need to appropriately change y.
- For example, to preserve the formula $d = v \cdot t$ – that the path is the product of speed and time:
  - when we change the unit for time,
  - we need to appropriately change the unit for speed.
- With this is mind, let us describe possible invariant dependencies.

8. Scaling-to-Scaling (sc-sc)

- Let us first consider the case when the dependence remains the same after we apply scaling to x and y.
- In precise terms, we assume that for every $\lambda > 0$, there exists a value $\mu(\lambda)$ (depending on λ) such that:
  - if $y = f(x)$,
  - then $Y = f(X)$, where $X = \lambda \cdot x$ and $Y = \mu(\lambda) \cdot y$.
- If we plug in the expressions for Y in terms of y and X in terms of x into $Y = f(X)$, we get $f(\lambda \cdot x) = \mu(\lambda) \cdot y$.
- Here, $y = f(x)$, so $f(\lambda \cdot x) = \mu(\lambda) \cdot f(x)$.
- It is known that every measurable dependence $f(x)$ with this property has the form $f(x) = A \cdot x^a$.

9. Comment

- The general proof is somewhat complicated.
- However, most physical dependencies are differentiable.
- For differentiable $f(x)$, this is easy to prove.
- Indeed, if $f(x)$ is differentiable, then the function $\mu(\lambda) = \dfrac{f(\lambda \cdot x)}{f(x)}$ is differentiable too.
- Thus, we can differentiate both sides of the equation $f(\lambda \cdot x) = \mu(\lambda) \cdot f(x)$ with respect to λ.
- As a result, we get $x \cdot f'(\lambda \cdot x) = \mu'(\lambda) \cdot f(x)$.
- In particular, for $\lambda = 1$, we get $x \cdot \dfrac{df}{dx} = a \cdot f$, where

$$a \stackrel{\text{def}}{=} \mu'(1).$$

10. Comment (cont-d)

- We can separate x and f if we multiply both sides of the equality by $\dfrac{dx}{x \cdot f}$: $\dfrac{df}{f} = a \cdot \dfrac{dx}{x}$.
- Integrating both sides, we get $\ln(f) = a \cdot \ln(x) + C$, where C is the integration constant.
- Applying the function $\exp(z)$ of both sides, we get the desired expression $f(x) = A \cdot x^a$, with $A = \exp(C)$.

11. Shift-to-Scaling (sh-sc)

- Let us consider the case when the dependence remains the same after we apply shift to x and scaling to y.
- In this case, for every x_0, there exists a value $\mu(x_0)$ (depending on x_0) such that:
  - if $y = f(x)$,
  - then we have $Y = f(X)$, where $X = x + x_0$ and

$$Y = \mu(x_0) \cdot y.$$

- If we plug in the expressions for Y in terms of y and X in terms of x into $Y = f(X)$, we get

$$f(x + x_0) = \mu(x_0) \cdot y.$$

- Here, $y = f(x)$, so $f(x + x_0) = \mu(x_0) \cdot f(x)$.
- It is known that every measurable dependence $f(x)$ with this property has the form $f(x) = A \cdot \exp(a \cdot x)$.

12. Comment

- If $f(x)$ is differentiable, then the function $\mu(x_0) = \dfrac{f(x+x_0)}{f(x)}$ is differentiable too.
- Thus, we can differentiate both sides of the equation $f(x+x_0) = \mu(x_0)\cdot f(x)$ with respect to x_0.
- As a result, we get $f'(x+x_0) = \mu'(x_0)\cdot f(x)$.
- For $x_0 = 0$, we get $\dfrac{df}{dx} = a \cdot f$, where $a \stackrel{\text{def}}{=} \mu'(0)$.
- We can separate the variables x and f if we multiply both sides of the equality by $\dfrac{dx}{f}$: $\dfrac{df}{f} = a \cdot dx$.
- Integrating both sides, we get $\ln(f) = a \cdot x + C$, where C is the integration constant.
- Applying the function $\exp(z)$ to both sides, we get $f(x) = A \cdot \exp(a \cdot x)$, with $A = \exp(C)$.

13. Scaling-to-Shift (sc-sh)

- Let us now consider the case when the dependence remains the same after we scale x and shift y.
- In precise terms, we assume that for every $\lambda > 0$, there exists a value $y_0(\lambda)$ (depending on λ) such that:
  - if $y = f(x)$,
  - then $Y = f(X)$, where $X = \lambda \cdot x$ and $Y = y + y_0(\lambda)$.
- If we plug in the expressions for Y in terms of y and X in terms of x $Y = f(X)$, we get $f(\lambda \cdot x) = y + y_0(\lambda)$.
- Here, $y = f(x)$, so $f(\lambda \cdot x) = f(x) + y_0(\lambda)$.
- It is known that every measurable dependence $f(x)$ with this property has the form $f(x) = a \cdot \ln(x) + C$.

14. Comment

- If $f(x)$ is differentiable, then the function $y_0(\lambda) = f(\lambda \cdot x) - f(x)$ is differentiable too.
- Thus, we can differentiate both sides of the equation $f(\lambda \cdot x) = f(x) + y_0(\lambda)$ with respect to λ.
- As a result, we get $x \cdot f'(\lambda \cdot x) = y_0'(\lambda)$.
- In particular, for $\lambda = 1$, we get $x \cdot \dfrac{df}{dx} = a$, where

$$a \stackrel{\text{def}}{=} y_0'(1).$$

- We can separate the variables x and f if we multiply both sides of the equality by $\dfrac{dx}{x}$: $df = a \cdot \dfrac{dx}{x}$.
- Integrating both sides, we get $f(x) = a \cdot \ln(x) + C$, where C is the integration constant.

15. Shift-to-Shift (sh-sh)

- In this case, for every x_0, there exists a value $y_0(x_0)$ such that:
  - if $y = f(x)$,
  - then we have $Y = f(X)$, where $X = x + x_0$ and

$$Y = y + y_0(x_0).$$

- If we plug in the expressions for Y in terms of y and X in terms of x into $Y = f(X)$, we get

$$f(x + x_0) = y + y_0(x_0).$$

- Here, $y = f(x)$, so $f(x + x_0) = f(x) + y_0(x_0)$.
- It is known that every measurable dependence $f(x)$ with this property has the form $f(x) = a \cdot x + C$.

16. Comment

- If $f(x)$ is differentiable, then the function $y_0(x_0) = f(x+x_0) - f(x)$ is differentiable too.
- Thus, we can differentiate both sides of the equation $f(x+x_0) = f(x) + y_0(x_0)$ with respect to x_0.
- As a result, we get $f'(x+x_0) = y_0'(x_0)$.
- In particular, for $x_0 = 0$, we get $f'(x) = a$, where

$$a \stackrel{\text{def}}{=} y_0'(0).$$

- Integrating, we get $f(x) = a \cdot x + C$, where C is the integration constant.

17. What We Want: A Brief Reminder

- We want to find the dependence of the quantity q (crack or pothole amount) on time t; we know:
  - that the for $t = 0$, the value $q(t)$ is small positive,
  - that the value $q(t)$ increases with time, and
  - that the value $q(t)$ tends to some large constant value when t increases.

18. What Are Possible Symmetries Here?

- For crack amount C and for pothole amount P, there is an absolute starting point: 0.
- Then, we have no cracks and no potholes.
- However, it makes sense to use different units of length and different units of area.
- So scaling makes perfect sense.
- For time, as we have mentioned, both shift and scaling make sense.

19. First Idea

- Let us see if any of the above symmetric dependencies satisfy the desired property.
- Since for q, only scaling makes sense, we can only consider two possibilities: sc-sc and sh-sc.
- Let us consider them one by one.
- In the sc-sc case, we have $q(t) = A \cdot t^a$.
- Since we want a non-negative value, we have $A > 0$.
- Since we want $q(t)$ to be increasing with time, we have to take $a > 0$.
- However, in this case:
  - $q(0)$ is zero – while we want it to be positive, and
  - $q(t)$ tends to infinity as t increases – while we want it to tend to some constant.

20. First Idea: sh-sc Case

- In the sh-sc case, we have $q(t) = A \cdot \exp(a \cdot t)$.
- Again, since we want a non-negative value, we have to take $A > 0$.
- Since we want $q(t)$ to be increasing with time, we have to take $a > 0$; in this case:
  - $q(0)$ is positive, which is exactly what we wanted,
  - however, $q(t)$ tends to infinity as t increases – while we want it to tend to some constant.

21. So What Do We Do?

- The first idea does not work, so what should we do?
- The above arguments about possible dependencies deal with the case when y directly depend on time t.
- However, in our case, cracks and potholes do not directly depend on time.
- What changes with time is stress, which, in its turn, causes the pavement to crack.
- In other words, instead of the direct dependence of the quantity q on time:
  - we have q depending on some auxiliary quantity z, and
  - we have z depending on time t.

22. So What Do We Do (cont-d)

- For both dependencies $q(z)$ and $z(t)$ we can have symmetry-motivated formulas.
- Let us see which combinations of these formulas provide the desired properties of $q(t) = q(z(t))$:
  - that this value is positive for $t = 0$,
  - that this value increases for $t > 0$, and
  - that this value tends to a finite limit when $t \to \infty$.

23. Possible Options of the $q(z)$ Dependence

- For q, only scaling is possible.
- So, for possible dependencies $q(z)$, we have:
  - either the sc-sc option $q(z) = A \cdot z^a$
  - or the sh-sc option $q(z) = A \cdot \exp(a \cdot z)$.
- In the sc-sc option $q(z) = A \cdot z^a$, it does not make sense to consider sh-sc or sc-sc options for $z(t)$; indeed:
  - as one can check, this will be equivalent to sh-sc or sc-sc symmetry for $q(t)$,
  - and we have already shown that this is not possible.
- So, to go beyond previously considered options, we need to consider two remaining options for $z(t)$:
  - sh-sh option $z(t) = a_1 \cdot t + C_1$, and
  - sc-sh option $z(t) = a_1 \cdot \ln(t) + C_1$.

24. Possible Options (cont-d)

- In the 1st case, $q(t) = A \cdot z^a = A \cdot (a_1 \cdot t + C_1)^a$, i.e., $q(t) = A_1 \cdot (t + c_2)^a$, where $A_1 = A \cdot (a_1)^a$ and $c_2 = \dfrac{C_1}{a_1}$.
- The need to have positive values of q implies $A > 0$, the need to have $q(t)$ increasing leads to $a > 0$.
- However then, for $t \to \infty$, the resulting expression tends to infinity – while we want it bounded.
- In the 2nd case, $q(t) = A \cdot (a_1 \cdot \ln(t) + C_1)^a$, i.e., $q(t) = A_1 \cdot (\ln(t) + c_2)^a$, with $A_1 = A \cdot (a_1)^a$ and $c_2 = \dfrac{C_1}{a_1}$.
- The need to have positive values of q implies $A > 0$, the need to have $q(t)$ increasing leads to $a > 0$.
- However then, for $t \to \infty$, the resulting expression also tends to infinity – while we want it bounded.

25. sh-sc Option $q(z) = A \cdot \exp(a \cdot z)$

- In this option, it does not make sense to consider sh-sh or sc-sh options for $z(t)$; indeed:
  - as one can check, this will be equivalent to sh-sc or sc-sc symmetry for $q(t)$,
  - and we have already shown that this is not possible.
- So, to go beyond previously considered options, we need to consider two remaining options for $z(t)$:
  - sc-sc option $z(t) = A_1 \cdot t^{a_1}$, and
  - sh-sc option $z(t) = A_1 \cdot \exp(a_1 \cdot t)$.
- In the 1st case, $q(t) = A \cdot \exp(a \cdot z) = A \cdot \exp((a \cdot A_1) \cdot t^{a_1})$.
- The need to have positive values of q implies $A > 0$.
- The behavior of this expression depends on the sign of the product $a \cdot A_1$.

26. sh-sc Option $q(z) = A \cdot \exp(a \cdot z)$ (cont-d)

- If $a \cdot A_1 > 0$, then the need to have $q(t)$ increasing leads to $a_1 > 0$.
- However then, for $t \to \infty$, the resulting expression tends to infinity – and we want it bounded.
- If $a \cdot A_1 < 0$, then the need to have $q(t)$ increasing leads to $a_1 < 0$.
- However then, for $t \to 0$, we have $t^{-|a_1|} \to \infty$, hence $(a \cdot A_1) \cdot t^{-|a_1|} \to -\infty$, and $q(t) = A \cdot \exp((a \cdot A_1) \cdot t^{-|a_1|}) \to 0$, but we want the value $q(0)$ to be positive.
- So, the only possible case is the second case, when

$$q(t) = A \cdot \exp(a \cdot z) = A \cdot ((a \cdot A_1) \cdot \exp(a_1 \cdot t)).$$

- This is exactly the desired formulas.
- Thus, we have indeed justified the empirical dependencies.

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