# Optimizing Trajectories for Unmanned Aerial Vehicles (UAVs) Patrolling the Border

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Patrolling the Border: . . .
How to Describe . . .
Constraints on . . .
Simplification of the . . .
Detection at Crossing . . .
Strategy Selected by . . .
Towards an Optimal . . .
An Optimal Strategy: . . .
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# 1. Patrolling the Border: a Practical Problem

- Remote areas of international borders are used by the adversaries: to smuggle drugs, to bring in weapons.
- It is therefore desirable to patrol the border, to minimize such actions.
- It is not possible to effectively man every single segment of the border.
- It is therefore necessary to rely on other types of surveillance.
- Unmanned Aerial Vehicles (UAVs):
  - from every location along the border, they provide an overview of a large area, and
  - they can move fast, w/o being slowed down by clogged roads or rough terrain.
- Question: what is the optimal trajectory for these UAVs?


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## 2. How to Describe Possible UAV Patrolling Strategies

- Let us assume that the time between two consequent overflies is smaller the time needed to cross the border.
- Ideally, such a UAV can detect all adversaries.
- In reality, a fast flying UAV can miss the adversary.
- We need to minimize the effect of this miss.
- The faster the UAV goes, the less time it looks, the more probable that it will miss the adversary.
- Thus, the velocity $v(x)$ is very important.
- By a patrolling strategy, we will mean a f-n $v(x)$ describing how fast the UAV flies at different locations $x$.


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## 3. Constraints on Possible Patrolling Strategies

1) The time between two consequent overflies should be smaller the time $T$ needed to cross the border:
   - the time during which a UAV passes from the location $x$ to the location $x+\Delta x$ is equal to $\Delta t = \dfrac{\Delta x}{v(x)}$;
   - thus, the overall flight time is equal to the sum of these times:
$$T = \int \frac{dx}{v(x)}.$$
2) UAV has the largest possible velocity $V$, so we must have $v(x) \le V$ for all $x$.

It is convenient to use the value $s(x) \stackrel{\text{def}}{=} \dfrac{1}{v(x)}$ called *slowness*, so

$$T = \int s(x)\,dx; \quad s(x) \ge S \left( \stackrel{\text{def}}{=} \frac{1}{V} \right).$$


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## 4. Simplification of the Constraints

- Since $s(x) \geq S$, the value $s(x)$ can be represented as $S + \Delta s(x)$, where $\Delta s(x) \stackrel{\text{def}}{=} s(x) - S$.
- The new unknown function satisfies the simpler constraint $\Delta s(x) \geq 0$.
- In terms of $\Delta s(x)$, the requirement that the overall time be equal to $T$ has a form $T = S \cdot L + \int \Delta s(x)\, dx$.
- This is equivalent to:

$$T_0 = \int \Delta s(x)\, dx, \text{ where:}$$

  - $L$ is the total length of the piece of the border that we are defending, and
  - $T_0 \stackrel{\text{def}}{=} T - S \cdot L$.


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## 5. Detection at Crossing Point $x$

- Let $h$ be the width of the border zone from which an adversary (A) is visible.
- Then, the UAV can potentially detect A during the time $h/v(x) = h \cdot s(x)$.
- So, the UAV takes $(h \cdot s(x))/\Delta t$ photos, where $\Delta t$ is the time per photo.
- Let $p_1$ be the probability that one photo misses A.
- It is reasonable to assume that different detection errors are independent.
- Then, the probability $p(x)$ that A is not detected is $p_1^{(h \cdot s(x))/\Delta t}$, i.e., $p(x) = \exp(-k \cdot s(x))$, where:

$$k \stackrel{\text{def}}{=} \frac{2h}{\Delta t} \cdot |\ln(p_1)|.$$


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## 6. Strategy Selected by the Adversary

- Let $w(x)$ denote the utility of the adversary succeeding in crossing the border at location $x$.
- Let us first assume that we know $w(x)$ for every $x$.
- According to decision theory, the adversary will select a location $x$ with the largest expected utility

$$u(x) = p(x) \cdot w(x) = \exp(-k \cdot s(x)) \cdot w(x).$$

- Thus, for each slowness function $s(x)$, the adversary's gain $G(s)$ is equal to

$$G(s) = \max_x u(x) = \max_x \left[\exp(-k \cdot s(x)) \cdot w(x)\right].$$

- We need to select a strategy $s(x)$ for which the gain $G(s)$ is the smallest possible.

$$G(s) = \max_x u(x) = \max_x \left[\exp(-k \cdot s(x)) \cdot w(x)\right] \to \min_{s(x)}.$$


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# 7. Towards an Optimal Strategy for Patrolling the Border

- Let $x_m$ be the location at which the utility $u(x) = \exp(-k \cdot s(x)) \cdot w(x)$ attains its largest possible value.
- If we have a point $x_0$ s.t. $u(x_0) < u(x_m)$ and $s(x_0) > S$:
  - we can slightly decrease the slowness $s(x_0)$ at the vicinity of $x_0$ (i.e., go faster in this vicinity) and
  - use the resulting time to slow down (i.e., to go slower) at all locations $x$ at which $u(x) = u(x_m)$.
- As a result, we slightly decrease the value

$$u(x_m) = \exp(-k \cdot s(x_m)) \cdot w(x_m).$$

- At $x_0$, we still have $u(x_0) < u(x_m)$.
- So, the overall gain $G(s)$ decreases.
- Thus, when the adversary's gain is minimized, we get

$$u(x) = u_0 = \text{const whenever } s(x) > S.$$


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## 8. Towards an Optimal Strategy (cont-d)

- *Reminder:* for the optimal strategy,

$$u(x) = w(x) \cdot \exp(-k \cdot s(x)) = u_0 \text{ whenever } s(x) > S.$$

- So, $\exp(-k \cdot s(x)) = \dfrac{u_0}{w(x)}$, hence

$$s(x) = \frac{1}{k} \cdot (\ln(w(x)) - \ln(u_0)) \text{ and } \Delta s(x) = \frac{1}{k} \cdot \ln(w(x)) - \Delta_0.$$

- Here, $\Delta_0 \stackrel{\text{def}}{=} \dfrac{1}{k} \cdot \ln(u_0) - S$.
- When $s(x)$ gets to $s(x) = S$ and $\Delta s(x) = 0$, we get $\Delta s(x) = 0$.
- Thus, we conclude that

$$\Delta s(x) = \max\left(\frac{1}{k} \cdot \ln(w(x)) - \Delta_0, 0\right).$$


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# 9. An Optimal Strategy: Algorithm

- *Reminder:* for some $\Delta_0$, the optimal strategy has the form

$$\Delta s(x) = \max\left(\frac{1}{k}\cdot\ln(w(x)) - \Delta_0, 0\right).$$

- *How to find* $\Delta_0$*:* from the condition that

$$\int \Delta s(x)\,dx =$$

$$\int \max\left(\frac{1}{k}\cdot\ln(w(x)) - \Delta_0, 0\right)\,dx = T_0.$$

- *Easy to check:* the above integral monotonically decreases with $\Delta_0$.
- *Conclusion:* we can use bisection to find the appropriate value $\Delta_0$.


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## 10. Taking Fuzzy Uncertainty into Account

- We assumed that we know the exact value of the adversary's gain $w(x)$ at different locations.
- In reality, we only have expert estimates for $w(x)$.
- To formalize these estimates, we can use fuzzy techniques.
- Once we have fuzzy $w(x)$, we can apply Zadeh's extension principle to the above crisp formulas.
- Thus, we come up with fuzzy recommendations about slowness, e.g., "go somewhat slow here", "go fast".
- It is well known that Zadeh's extension principle is equivalent to processing $\alpha$-cuts:

$$Y(\alpha) = f(X_1(\alpha), \ldots, X_n(\alpha)) =$$

$$\{f(x_1, \ldots, x_n) : x_1 \in X_1(\alpha), \ldots, x_n \in X_n(\alpha)\}.$$

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## 11. Fuzzy Case (cont-d)

- *Reminder:* $Y(\alpha) = f(X_1(\alpha), \ldots, X_n(\alpha)) =$

$$\{f(x_1, \ldots, x_n) : x_1 \in X_1(\alpha), \ldots, x_n \in X_n(\alpha)\}.$$

- *Important case:* the function $y = f(x_1, \ldots, x_n)$ is:
  - an increasing function of the variables $x_1, \ldots, x_k$ and
  - a decreasing function of the variables $x_{k+1}, \ldots, x_n$).
- *In this case:* the $\alpha$-cut has the form $[\underline{Y}(\alpha), \overline{Y}(\alpha)]$, where

$$\underline{Y}(\alpha) = f(\underline{X}_1(\alpha), \ldots, \underline{X}_k(\alpha), \overline{X}_{k+1}(\alpha), \ldots, \overline{X}_n(\alpha));$$

$$\overline{Y}(\alpha) = f(\overline{X}_1(\alpha), \ldots, \overline{X}_k(\alpha), \underline{X}_{k+1}(\alpha), \ldots, \overline{X}_n(\alpha)).$$

- *Our case:* $\Delta_0$ increases with each value $w(x)$.
- *Conclusion:* efficient algorithm (see paper for details).


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