How to Describe Measurement Uncertainty and Uncertainty of Expert Estimates?

Nicolas Madrid and Irina Perfilieva
Institute of Research and Applications of Fuzzy Modeling
University of Ostrava, Ostrava, Czech Republic
nicolas.madrid@osu.cz, Irina.Perfilieva@osu.cz

Vladik Kreinovich
Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
vladik@utep.edu

1. How Can We Describe Measurement Uncertainty: Formulation of the Problem

- We want to know the actual values of different quantities.
- To get these values, we perform measurements.
- Measurements are never absolutely accurate.
- The actual value $A(u)$ of the corr. quantity is, in general, different from the measurement result $M(u)$.
- It is therefore desirable to describe what are the possible values of $A(u)$.
- This will be a perfect way to describe uncertainty:
  - for each measurement result $M(u)$,
  - we describe the set of all possible values of $A(u)$.
- How can we attain this description?

2. In Practice, We Don't Know Actual Values

- Ideally, for diff. situations u, we should compare the measurement result $M(u)$ with the actual value $A(u)$.
- The problem is that we do not know the actual value.
- A usual approach is to compare
  - the measurement result $M(u)$ with
  - the result $S(u)$ of measuring the same quantity by a much more accurate ("standard") MI.
- From this viewpoint, the above problem can be reformulated as follows:
  - we know the measurement result $M(u)$ corresponding to some situation u,
  - we want to find the set of possible values $S(u)$ that we would have obtained if we apply a standard MI.

3. Case of Absolute Measurement Error

- In some cases, we know the upper bound Δ on the absolute value of the measurement error $M(u) - A(u)$:

$$|M(u) - A(u)| \le \Delta.$$

- In this case, once we know the measurement result $M(u)$, we can conclude that

$$M(u) - \Delta \le A(u) \le M(u) + \Delta.$$

- In more general terms, we can describe the corresponding bounds as $f(M(u)) \le A(u) \le g(M(u))$, where

$$f(x) \stackrel{\text{def}}{=} x - \Delta \text{ and } g(x) \stackrel{\text{def}}{=} x + \Delta.$$

4. Case of Relative Measurement Error

- In some other cases, we know the upper bound δ on the *relative* measurement error:

$$\frac{|M(u)-A(u)|}{|A(u)|} \le \delta.$$

- In this case, for positive values,

$$(1-\delta)\cdot A(u) \le M(u) \le (1+\delta)\cdot A(u).$$

- Thus, once we know the measurement result $M(u)$, we can conclude that

$$\frac{M(u)}{1+\delta} \le A(u) \le \frac{M(u)}{1-\delta}.$$

- So, we have $f(M(u)) \le A(u) \le g(M(u))$ for

$$f(x) \stackrel{\text{def}}{=} \frac{x}{1+\delta} \text{ and } g(x) \stackrel{\text{def}}{=} \frac{x}{1-\delta}.$$

5. In Some Cases, We Have Both Types of Measurement Errors

- In some cases, we have both *additive* (absolute) and *multiplicative* (relative) measurement errors:

$$A(u) - \Delta - \delta \cdot A(u) \le M(u) \le A(u) + \Delta + \delta \cdot A(u).$$

- In this case:

$$\frac{M(u) - \Delta}{1 + \delta} \le A(u) \le \frac{M(u) + \Delta}{1 - \delta}.$$

- So, we have $f(M(u)) \le A(u) \le g(M(u))$, where:

$$f(x) \stackrel{\text{def}}{=} \frac{x - \Delta}{1 + \delta} \text{ and } f(x) \stackrel{\text{def}}{=} \frac{x + \Delta}{1 - \delta}.$$

6. Towards a General Case

- The above formulas assume that the measurement accuracy is the same for the whole range.
- In reality, measuring instruments have different accuracies Δ and δ in different ranges.
- Hence, $f(x)$ and $g(x)$ are non-linear.
- When $M(u)$ is larger, this means that the bounds on possible values of $A(u)$ increase (or do not decrease).
- Thus, $f(x)$ and $g(x)$ are monotonic.
- To describe the accuracy of a general measuring instrument, it is therefore reasonable to use:
  - the largest of the monotonic functions $f(x)$ for which $f(M(u)) \leq A(u)$ and
  - the smallest of the monotonic functions $g(x)$ for which $A(u) \leq g(M(u))$.

7. From Measurements to Expert Estimates

- In areas such as medicine, expert estimates are very important.
- Expert estimates often result in "values" from a partially ordered set.
- Examples: "somewhat probable", "very probable", etc.
- Such possibilities are described in different generalizations of the traditional $[0,1]$-based fuzzy logic.
- In all such extensions, there is *order* (sometimes partial) on the corresponding set of value L:

$$\ell < \ell' \text{ means that } \ell' \text{ represents a stronger expert's degree of confidence than } \ell.$$

8. Need to Describe Uncertainty of Expert Estimates

- Some experts are very good, in the sense that based on their estimates $S(u)$, we make very effective decisions.
- Other experts may be less accurate.
- It is therefore desirable to gauge the uncertainty of such experts in relation to the "standard" (very good) ones.
- To make a good decision based on the expert's estimate $M(u)$, we need to produce bounds on $S(u)$:

$$f(M(u)) \le S(u) \le g(M(u)).$$

- It is thus desirable to find:
  - the largest of the monotonic functions $f(x)$ for which $f(M(u)) \le S(u)$ and
  - the smallest of the monotonic functions $g(x)$ for which $S(u) \le g(M(u))$.

9. What Is Known and What We Do in This Talk

- When $L = [0, 1]$, the existence of the largest $f(x)$ and smallest $g(x)$ is already known.
- We analyze for which partially ordered sets such largest $f(x)$ and smallest $g(x)$ exist.
- It turns out that they exist for complete lattices.
- In general, they do not exist for more general partially ordered sets.
- To be more precise,
  - the largest $f(x)$ always exists only for complete lower semi-lattices (definitions given later), while
  - the smallest $g(x)$ always exists only for complete upper semi-lattices.

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10. Main Result about f

- By $\mathcal{F}(F, G)$, we denote the set of all monotonic functions f for which $f(F(u)) \le G(u)$ for all $u \in U$.
- An ordered set is called a *complete lower semi-lattice* if for every set S:
  - among all its lower bounds,
  - there exists the largest one.
- **Theorem.** *For an ordered set L, the following two conditions are equivalent to each other:*
  - *L is a complete lower semi-lattice;*
  - *for every two functions $F, G : U \to L$, the set $\mathcal{F}(F, G)$ has the largest element.*

11. Main Result about g

- By $\mathcal{G}(F,G)$, we denote the set of all monotonic functions g for which $F(u) \le g(G(u))$ for all $u \in U$.
- An ordered set is called a *complete upper semi-lattice* if for every set S:
  - among all its upper bounds,
  - there exists the smallest one.
- **Theorem.** *For an ordered set L, the following two conditions are equivalent to each other:*
  - *L is a complete upper semi-lattice;*
  - *for every two functions $F,G : U \to L$, the set $\mathcal{G}(F,G)$ has the smallest element.*

12. What If There Is No Bias?

- In some practical situations, measuring instrument has a *bias* (shift):
  - a clock can be regularly 2 minutes behind,
  - a thermometer can regularly show temperatures which are 3 degrees higher, etc.
- Bias means that we get the measurement result $M(u)$ *cannot* be equal to the actual value $A(u)$.
- Bias can easily be eliminated by re-calibrating the measuring instrument.
- For example, if I move to a different time zone, I can simply add the corresponding time difference.
- It is thus reasonable to assume that the bias has already been eliminated.
- So $A(u) = M(u)$ is one of the possible actual values.

13. What If There Is No Bias? (cont-d)

- It is reasonable to assume that $A(u) = M(u)$ is one of the possible actual values.
- For this value $A(u) = M(u)$, our inequality $f(M(u)) \le A(u) \le g(M(u))$ implies that

$$f(x) \le x \le g(x).$$

- So, it makes sense to only consider functions $f(x)$ and $g(x)$ for which $f(x) \le x$ and $x \le g(x)$.
- It turns out that similar results hold when we thus restrict the functions $f(x)$ and $g(x)$.

14. No-Bias Results

- Let $\mathcal{F}_u(F,G)$ be the set of all monotonic $f(x)$ s.t.:
  - $f(x) \le x$ and
  - $f(F(u)) \le G(u)$ for all u.
- **Theorem.** *If L is a complete lower semi-lattice, then:*
  - *for every two functions $F, G : U \to L$,*
  - *the set $\mathcal{F}_u(F,G)$ has the largest element.*
- Let $\mathcal{G}_u(F,G)$ be the set of all monotonic functions $g(x)$ s.t.:
  - $x \le g(x)$ and
  - $F(u) \le g(G(u))$ for all u.
- **Theorem.** *If L is a complete upper semi-lattice, then:*
  - *for every two functions $F, G : U \to L$,*
  - *the set $\mathcal{G}_u(F,G)$ has the smallest element.*

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16. Proof that the Largest f Exists for Complete Lower Semi-Lattices

- We will prove that the desired function is

$$f_{F,G}(x) \stackrel{\text{def}}{=} \bigwedge \{G(u) : x \le F(u)\}.$$

- In other words, we will prove:
  - that $f_{F,G}$ belongs to the class $\mathcal{F}(F,G)$, and
  - that $f_{F,G}$ is the largest function in this class.
- Let us first prove that $f_{F,G} \in \mathcal{F}(F,G)$, i.e., that for every u, we have $f_{F,G}(F(u)) \le G(u)$.
- Indeed, for $x = F(u)$, we have $x \le F(u)$, and thus, $G(u)$ belongs to the set $S_0 \stackrel{\text{def}}{=} \{G(u) : x \le F(u)\}$.
- Thus, $G(u)$ is larger than or equal to the largest lower bound $f_{F,G}(x) = \bigwedge \{G(u) : x \le F(u)\}$ of S_0:

$$f_{F,G}(F(u)) = f_{F,G}(x) \le G(u).$$

17. Proof (cont-d)

- Let us now prove that $f_{F,G}$ is the largest in the class $\mathcal{F}(F,G)$: if $f \le \mathcal{F}(F,G)$, then $f \le f_{F,G}$.
- Indeed, let $f \in \mathcal{F}(F,G)$.
- By definition of this class, this means that f is monotonic and $f(F(u)) \le G(u)$ for all u.
- Let us pick some $x \in L$ and show that $f(x) \le f_{F,G}(x)$.
- Indeed, for every value $u \in U$ for which $x \le F(u)$, we have, due to monotonicity, $f(x) \le f(F(u))$.
- Since $f(F(u)) \le G(u)$, we conclude that $f(x) \le G(u)$.
- So, the value $f(x)$ is smaller than or equal to all elements of the set $S_0 = \{G(u) : x \le F(u)\}$.
- Thus, $f(x)$ is a lower bound for S_0.

18. Proof (cont-d)

- Every lower bound is smaller than or equal to the largest lower bound

$$f_{F,G}(x) = \bigwedge \{G(u) : x \le F(u)\}.$$

- So indeed $f(x) \le f_{F,G}(x)$.
- Let us now prove that $\mathcal{F}(F,G) = \{f \in M_L : f \le f_{F,G}\}$.
- We have shown that every function $f \in \mathcal{F}(F,G)$ is $\le f_{F,G}$, i.e., that

$$\mathcal{F}(F,G) \subseteq \{f \in M_L : f \le f_{F,G}\}.$$

- Vice versa, if $f \le f_{F,G}$, then for every u,
  - from $f_{F,G}(F(u)) \le G(u)$ and $f(F(u)) \le f_{F,G}(F(u))$,
  - we conclude that $f(F(u)) \le G(u)$, i.e., that indeed $f \in \mathcal{F}(F,G)$. The statement is proven.

19. Proof that Only Complete Lower Semi-Lattices Have This Property

- Let us assume that the ordered set L has the above property.
- Let us prove that L is a complete lower semi-lattice.
- Indeed, let $S \subseteq L$ be any subset of L.
- Let us take $U = S$, and take $G(u) = u$ for all $u \in S$.
- Let us also pick any element $x_0 \in L$ and take $F(u) = x_0$ for all $u \in S$.
- Because of our assumption, the set $\mathcal{F}(F, G)$ of all $f(x)$ s.t. $f(F(u)) \le G(u)$ for all u has the largest element.
- Because of our choice of $F(u)$ and $G(u)$, $f(F(u)) \le G(u)$ simply means that $f(x_0) \le u$ for all $u \in S$.
- So, $f(F(u)) \le G(u)$ means that $f(x_0)$ is the lower bound for the set S.

20. Proof (cont-d)

- Our assumption implies that there is the largest among all the functions $f \in \mathcal{F}(F, G)$.
- Thus, there is the largest among all the lower bounds for the set S.
- This is exactly the definition of the complete lower semi-lattice.
- The statement is proven.