Why Ricker Wavelets Are Successful in Processing Seismic Data: Towards a Theoretical Explanation

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1. Seismic Waves: Brief Reminder

- Already ancient scientists noticed that earthquakes generate waves which can be detected at large distances.
- These waves were called *seismic waves*, after the Greek word "seismos" meaning an earthquake.
- After a while, scientists realized that from the seismic waves, we can extract:
  - not only important information about earthquakes,
  - but also information about the media through which these waves propagate.
- Different layers reflect, refract, and/or delay signals differently.
- So, by observing the coming waves, we can extract a lot of information about these layers.

2. Seismic Data is Very Useful

- Since earthquakes are rare, geophysicists set up small explosions to get seismic waves.
- The resulting *seismic* information helps:
  - geophysicists, petroleum and mining engineers, to find mineral deposits;
  - hydrologists to find underground water reservoirs;
  - civil engineers to get check stability of the underground layers below a future building, etc.
- In particular, computational intelligence techniques are actively used in processing seismic data.

3. Ricker Wavelets: Reminder

- We need to describe how the amplitude $x(t)$ of a seismic signal changes with time t.
- In 1953, N. Ricker proposed to use a linear combination of wavelets of the type

$$x_0(t) = \left(1 - \frac{(t-t_0)^2}{\sigma^2}\right) \cdot \exp\left(-\frac{(t-t_0)^2}{2\sigma^2}\right).$$

- Different wavelets correspond to:
  - different moments of time t_0 and
  - different values of the parameter σ describing the duration of this wavelet signal.
- The power spectrum $S(\omega)$ of this wavelet has the form

$$S(\omega) = K \cdot \omega^2 \cdot \exp(-c \cdot \omega^2), \text{ where } c = \sigma^2.$$

4. Ricker Wavelets Are Empirically Successful

- Since the original Ricker's paper, Ricker wavelets have been successfully used in processing seismic signals.
- In particular, Ricker wavelets are used with computational intelligence techniques,
- The power spectrum $S(\omega)$ of the seismic signal is represented as a linear combination of Ricker spectra:

$$S(\omega) \approx \sum_{i=1}^{n} K_i \cdot \omega^2 \cdot \exp(-c_i \cdot \omega^2).$$

- This description requires $2n$ parameters K_i and c_i.
- Often, this approximation of the most accurate of all approximations with the fixed number of parameters.

5. Need for a Theoretical Explanation

- *Empirical fact:* Ricker wavelets, in general, lead to a better approximation of the seismic spectra.
- *Problem:*
  - there are many possible families of approximating functions, and
  - only few of these families were actually tested.
- *Natural question:*
  - are Ricker wavelets indeed the best or
  - they are just a good approximation to some even better family of approximating functions?
- *What we show:* Ricker wavelets are the best.

6. How Each Propagation Layer Affects the Seismic Signal

- Layers are not homogeneous; as a result:
  - the same seismic signal, when passing through different locations on the same layer,
  - can experience different time delays.
- Thus, a unit pulse signal at moment 0 is transformed into a signal $m(t)$ which is distributed in time.
- A signal $x(t)$ can be represented as a linear combination of pulses of amplitudes $x(s_i)$ at moments s_i.
- Each such pulse is transformed into $m(t-s_i)\cdot x(s_i)$.
- So, each layer transforms the original signal $x(t)$ into the new signal $\int m(t-s)\cdot x(s)\,ds$.

7. What is the Joint Effect of Propagating the Signal Through Several Layers?

- A signal $x_0(t)$ passes through the first layer, and is thus transformed into $x_1(t) = \int m_1(t-s) \cdot x_0(s)\,ds$.
- The signal $x_1(t)$ passes through the second layer, and is, thus, transformed into $y(t) = \int m_2(t-s) \cdot x_1(s)\,ds$.
- Substituting the expression for $x_1(s)$, we conclude that $y(t) = \int m(t-u) \cdot x_0(u)\,du$, where

$$m(t) = \int m_1(s) \cdot m_2(t-s)\,ds.$$

- The formula is known as the *convolution* of two functions $m_1(t)$ and $m_2(s)$ corresponding to the two layers.
- In general, the joint effect of several layers is a convolution of several functions $m_i(t)$.

8. How to Describe Convolutions of Several Functions?

- A similar problem appears in probability theory.
- If independent random variables x_i have pdf's $\rho_i(x_i)$, then the pdf $\rho(x)$ of $x = \sum x_i$ is a convolution:

$$\rho(x) = \int \rho_1(x_1) \cdot \rho_2(x - x_1)\, dx_1.$$

- According to the Central Limit Theorem:
  - if we have a large number of small independent random variables,
  - then the distribution for their sum is close to Gaussian (normal).
- Different layers are independent.
- Thus, the joint effect of several layers is described by the Gaussian formula $m(t) = C \cdot \exp\left(-\frac{t^2}{2\sigma^2}\right)$.

9. Fourier Transform Helps

- We know that $y(t) = \int m(t-s)\cdot x(s)\,ds$.
- If we know n values of $m(t)$ and $x(t)$, we need n^2 computations to follow this formula.
- FFT computes Fourier transform

$$\hat{f}(\omega) \stackrel{\text{def}}{=} \int \exp(-\mathrm{i}\cdot\omega\cdot x)\cdot f(x)\,dx \text{ in time } O(n\cdot\ln n) \ll n^2.$$

- In terms of Fourier transforms, $\hat{y}(\omega) = \hat{m}(\omega)\cdot\hat{x}(\omega)$.
- For Gaussian $m(t)$, its Fourier transform is also Gaussian, so $\hat{y}(\omega) = \text{const}\cdot\exp\left(-\frac{1}{2}\cdot\sigma^2\cdot\omega^2\right)\cdot\hat{x}(\omega)$.
- We are interested in the power spectra $X(\omega) \stackrel{\text{def}}{=} |\hat{x}(\omega)|^2$ and $Y(\omega) \stackrel{\text{def}}{=} |\hat{y}(\omega)|^2$, so

$$Y(\omega) = \text{const}\cdot\exp(-\alpha\cdot\omega^2)\cdot X(\omega),\ \text{where } \alpha \stackrel{\text{def}}{=} \sigma^2.$$

10. Analysis of the Problem

- We want to select a function $F(\omega)$ that describes observed power spectrum of the seismic signal $x(t)$.
- By definition, power spectrum $X(\omega)$ is always non-negative, so we require that $F(\omega) \geq 0$.
- A single seismic signal quickly fades with time.
- It is known that when a signal $x(t)$ is limited in time, its Fourier transform $\hat{x}(\omega)$ is differentiable.
- So, we require that $F(\omega)$ be smooth.
- Thus, its power spectrum $X(\omega) = \hat{x}(\omega) \cdot (\hat{x}(\omega))^*$, where z^* means complex conjugation, is also differentiable.
- A seismic signal can have different amplitude: if $x(t)$ is a reasonable signal, then $C \cdot x(t)$ is also reasonable.
- If $F(\omega)$ is a good approximation to spectrum $X(\omega)$, then for $K \cdot X(\omega)$, it is reasonable to use $K \cdot F(\omega)$.

11. Analysis of the Problem

- So, we look for an approximating family $\{K \cdot F(\omega)\}_K$.
- Some seismic events are aster, some slower:
  - if $x(t)$ is a reasonable seismic signal,
  - then $x(t/c)$ is also reasonable.
- For $x(t/c)$, the spectrum is $F(c \cdot \omega)$.
- Thus, we look for a family $\{K \cdot F(c \cdot \omega)\}_{K,c}$.
- We want to approximate observed energy spectra $Y_i(\omega) = \text{const} \cdot \exp(-\alpha_i \cdot \omega^2) \cdot X(\omega)$; when $\alpha_1 < \alpha_2$:

$$Y_2(\omega) = \exp(-\alpha \cdot \omega^2) \cdot Y_1(\omega), \text{ where } \alpha \stackrel{\text{def}}{=} \alpha_2 - \alpha_1.$$

- So, if $X(\omega)$ is a reasonable power spectrum, then the function $\exp(-\alpha \cdot \omega^2) \cdot X(\omega)$ is also reasonable.
- It is thus reasonable to require that $\exp(-\alpha \cdot \omega^2) \cdot F(\omega)$ have the form $K \cdot F(c \cdot \omega)$ for some K and c.

12. Main Result

- Let $F(\omega) \geq 0$ be infinitely differentiable.
- We say that a family $\{K \cdot F(c \cdot \omega)\}_{K,c}$ is *propagation-invariant* if for every α, there exist $K(\alpha)$ and $c(\alpha)$ s.t.

$$\exp(-\alpha \cdot \omega^2) \cdot F(\omega) = K(\alpha) \cdot F(c(\alpha) \cdot \omega).$$

- *Every propagation-invariant family corresponds to*

$$F(\omega) = \omega^{2n} \cdot \exp(-\omega^2) \textit{ for some } n = 0, 1, \ldots$$

- The simplest case $n = 0$ correspond to a propagation of a simple pulse.
- Thus, the case $n = 0$ does not reflect the shape of the original signal.
- The simplest non-trivial case is $n = 1$, which is exactly the Ricker wavelet.

13. Conclusions

- A natural way to process dynamic signals is to approximate them by functions from an appropriate family.
- In this paper, we consider the problem of processing seismic data.
- For this problem, we formulated reasonable requirements for approximating functions.
- We showed that the simplest family of functions satisfying these requirements is the family of Ricker wavelets.
- This theoretical result is in good accordance with empirical findings: that in many cases,
  - for a given accuracy,
  - the use of Ricker wavelets enables us to use fewer parameters.

14. Future Work

- In many cases, Ricker wavelet provide a very good approximation for seismic data.
- However, sometimes, the approximation quality of Ricker wavelets needs improvement.
- Thus, it is not always sufficient to use the simplest possible approximate family of functions.
- More complex approximating functions are sometimes needed.
- It is therefore desirable:
  - to find the best of such *more complex* approximating families,
  - similar to how we found that the best of the *simplest* approximating families consists of Ricker wavelets.

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16. Proof

- We require that $\exp(-\alpha\cdot\omega^2)\cdot F(\omega) = K(\alpha)\cdot F(c(\alpha)\cdot\omega)$.
- We know that all the functions $F(\omega)$, $K(\alpha)$, and $c(\alpha)$ are differentiable.
- Thus, we can differentiate the above equality, and get

$$-F(\omega)\cdot\exp(-\alpha\cdot\omega^2)\cdot\omega^2 =$$

$$K'(\alpha)\cdot F(c(\alpha)\cdot\omega) + K(\alpha)\cdot F'(c(\alpha)\cdot\omega)\cdot c'(\alpha)\cdot\omega.$$

- For $\alpha = 0$, we use $K(0) = c(0) = 1$ to get

$$-F(\omega)\cdot\omega^2 = k\cdot F(\omega) + F'(\omega)\cdot c\cdot\omega, \text{ where } k \stackrel{\text{def}}{=} K'(0),\ \ c \stackrel{\text{def}}{=} c'(0).$$

- Moving all terms $\sim F(\omega)$ to the left-hand side, we get

$$F\cdot(-k-\omega^2) = c\cdot\frac{dF}{d\omega}\cdot\omega.$$

17. Proof (cont-d)

- Let us move all the terms dF and F to the right-hand side and all the other terms to the left-hand side:

$$\frac{1}{c}\cdot\frac{-k-\omega^2}{\omega}=\frac{dF}{F},\ \text{i.e.,}\ -\frac{k}{c}\cdot\frac{1}{\omega}-c\cdot\omega=\frac{dF}{F}.$$

- Integrating both sides, we get

$$C-\frac{k}{c}\cdot\ln(\omega)-\frac{c}{2}\cdot\omega^2=\ln(F).$$

- By exponentiating both sides, we conclude that

$$F(\omega)=A\cdot\omega^b\cdot\exp(-B\cdot\sigma^2),\ \text{w}/A=\exp(C),\ b=-\frac{k}{c},\ B=\frac{c}{2}.$$

- The requirement that $F(\omega)$ is infinitely differentiable for $\omega=0$ implies that b is a natural number.

- The requirement that $F(\omega)\ge 0$ means that b is even: $b=2n$ for some natural number n. Q.E.D.