Decision Making under Interval (and More General) Uncertainty: Monetary vs. Utility Approaches

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1. Outline

- Need for decision making under uncertainty
- Monetary approach: interval, probabilistic, fuzzy cases
- Utility-based approach
- Group decision making.
- Case study: selecting a location for a meteorological tower.

Part I
Need for Decision Making under Uncertainty

2. Need for Decision Making

- In many practical situations:
  - we have several alternatives, and
  - we need to select one of these alternatives.
- *Examples:*
  - a person saving for retirement needs to find the best way to invest money;
  - a company needs to select a location for its new plant;
  - a designer must select one of several possible designs for a new airplane;
  - a medical doctor needs to select a treatment for a patient.

3. Need for Decision Making Under Uncertainty

- Decision making is easier if we know the exact consequences of each alternative selection.
- Often, however:
  - we only have an incomplete information about consequences of different alternative, and
  - we need to select an alternative under this uncertainty.

4. Real-Life Examples

- Decision theory was used to select a location of the Mexico City airport.
- On the one hand, the closer the airport to the city, the better.
- However, Mexico City is surrounded by mountains, including a volcano (Popo).
- Sometimes, the visibility is very low for flying.
- So, another option is to build is outside the valley.
- Then, we will be able to fly all the time.
- But the disadvantage is it will be far from the city.
- Additional aspect: we plan for the future, and future is uncertain.
- In the 1970s, Mexico asked specialists in decision theory to help.

5. Real-Life Examples (cont-d)

- Decision was made to build it in the city.
- It is very convenient: a metro line goes directly to it.
- And yes, sometimes flights are canceled.
- Another example: a network of radiotelescopes for VLBI.
- We want to be able to provide the best resolution for the objects.
- Problem: we do not know what we will see.
- The whole purpose of the network is to find new objects, beyond what we saw before.
- Third example: selecting a landing place for the first Moon landing.
- Uncertainty: we do not know the properties of the Lunar soil.

Part II
Monetary Approach: Interval, Probabilistic, Fuzzy Cases

6. When Monetary Approach Is Appropriate

- In many situations, e.g., in financial and economic decision making, the decision results:
  - either in a money gain (or loss) and/or
  - in the gain of goods that can be exchanged for money or for other goods.
- In this case, we select an alternative which the highest exchange value, i.e., the highest price u.
- Uncertainty means that we do not know the exact prices.
- The simplest case is when we only know lower and upper bounds on the price: $u \in [\underline{u}, \overline{u}]$.

7. Hurwicz Optimism-Pessimism Approach to Decision Making under Interval Uncertainty

- L. Hurwicz's idea is to select an alternative s.t.

$$\alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u} \to \max .$$

- Here, $\alpha_H \in [0, 1]$ described the optimism level of a decision maker:
  - $\alpha_H = 1$ means optimism;
  - $\alpha_H = 0$ means pessimism;
  - $0 < \alpha_H < 1$ combines optimism and pessimism.

\+ This approach works well in practice.

– However, this is a semi-heuristic idea.

? It is desirable to come up with an approach which can be uniquely determined based first principles.

8. Numerical Example

- Suppose that we have two alternatives:
  - one in which we gain \$1,000 for sure, and
  - one in which we may gain \$2,500, but may gain nothing, and
  - we have no information about the probabilities of different gains.
- Which option should we choose?
- An optimist chooses the second alternative.
- A pessimist chooses the first alternative.
- For $\alpha = 0.5$, the second alternative is better:

$$\alpha \cdot \overline{u} + (1 - \alpha) \cdot \underline{u} = 0.5 \cdot 2500 + 0.5 \cdot 0 = 1250 > 1000.$$

- In general, for $\alpha > 0.4$, the second alternative is better, otherwise the first one.

9. Fair Price Approach: An Idea

- When we have a full information about an object, then:
  - we can express our desirability of each possible situation
  - by declaring a price that we are willing to pay to get involved in this situation.
- Once these prices are set, we simply select the alternative for which the participation price is the highest.
- In decision making under uncertainty, it is not easy to come up with a fair price.
- A natural idea is to develop techniques for producing such fair prices.
- These prices can then be used in decision making, to select an appropriate alternative.

10. Case of Interval Uncertainty

- *Ideal case:* we know the exact gain u of selecting an alternative.
- *A more realistic case:* we only know the lower bound $\underline{u}$ and the upper bound $\overline{u}$ on this gain.
- *Comment:* we do not know which values $u \in [\underline{u}, \overline{u}]$ are more probable or less probable.
- This situation is known as *interval uncertainty*.
- We want to assign, to each interval $[\underline{u}, \overline{u}]$, a number $P([\underline{u}, \overline{u}])$ describing the fair price of this interval.
- Since we know that $u \le \overline{u}$, we have $P([\underline{u}, \overline{u}]) \le \overline{u}$.
- Since we know that $\underline{u} \le u$, we have $\underline{u} \le P([\underline{u}, \overline{u}])$.

11. Case of Interval Uncertainty: Monotonicity

- *Case 1:* we keep the lower endpoint $\underline{u}$ intact but increase the upper bound.
- This means that we:
  - keeping all the previous possibilities, but
  - we allow new possibilities, with a higher gain.
- In this case, it is reasonable to require that the corresponding price not decrease:

$$\text{if } \underline{u} = \underline{v} \text{ and } \overline{u} < \overline{v} \text{ then } P([\underline{u}, \overline{u}]) \le P([\underline{v}, \overline{v}]).$$

- *Case 2:* we dismiss some low-gain alternatives.
- This should increase (or at least not decrease) the fair price:

$$\text{if } \underline{u} < \underline{v} \text{ and } \overline{u} = \overline{v} \text{ then } P([\underline{u}, \overline{u}]) \le P([\underline{v}, \overline{v}]).$$

12. Additivity: Idea

- Let us consider the situation when we have two consequent independent decisions.
- We can consider two decision processes separately.
- We can also consider a single decision process in which we select a pair of alternatives:
  - the 1st alternative corr. to the 1st decision, and
  - the 2nd alternative corr. to the 2nd decision.
- If we are willing to pay:
  - the amount u to participate in the first process, and
  - the amount v to participate in the second decision process,
- then we should be willing to pay $u + v$ to participate in both decision processes.

13. Additivity: Case of Interval Uncertainty

- About the gain u from the first alternative, we only know that this (unknown) gain is in $[\underline{u}, \overline{u}]$.
- About the gain v from the second alternative, we only know that this gain belongs to the interval $[\underline{v}, \overline{v}]$.
- The overall gain $u + v$ can thus take any value from the interval

$$[\underline{u}, \overline{u}] + [\underline{v}, \overline{v}] \stackrel{\text{def}}{=} \{u + v : u \in [\underline{u}, \overline{u}], v \in [\underline{v}, \overline{v}]\}.$$

- It is easy to check that

$$[\underline{u}, \overline{u}] + [\underline{v}, \overline{v}] = [\underline{u} + \underline{v}, \overline{u} + \overline{v}].$$

- Thus, the additivity requirement about the fair prices takes the form

$$P([\underline{u} + \underline{v}, \overline{u} + \overline{v}]) = P([\underline{u}, \overline{u}]) + P([\underline{v}, \overline{v}]).$$

14. Fair Price Under Interval Uncertainty

- By a *fair price under interval uncertainty*, we mean a function $P([\underline{u}, \overline{u}])$ for which:
  - $\underline{u} \le P([\underline{u}, \overline{u}]) \le \overline{u}$ for all $\underline{u}$ and $\overline{u}$ (*conservativeness*);
  - if $\underline{u} = \underline{v}$ and $\overline{u} < \overline{v}$, then $P([\underline{u}, \overline{u}]) \le P([\underline{v}, \overline{v}])$ (*monotonicity*);
  - (*additivity*) for all $\underline{u}$, $\overline{u}$, $\underline{v}$, and $\overline{v}$, we have

$$P([\underline{u} + \underline{v}, \overline{u} + \overline{v}]) = P([\underline{u}, \overline{u}]) + P([\underline{v}, \overline{v}]).$$

- *Theorem:* Each fair price under interval uncertainty has the form

$$P([\underline{u}, \overline{u}]) = \alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u} \text{ for some } \alpha_H \in [0, 1].$$

- *Comment:* we thus get a new justification of Hurwicz optimism-pessimism criterion.

15. Proof: Main Ideas

- Due to monotonicity, $P([u,u]) = u$.
- Due to monotonicity, $\alpha_H \stackrel{\text{def}}{=} P([0,1]) \in [0,1]$.
- For $[0,1] = [0,1/n] + \ldots + [0,1/n]$ (n times), additivity implies $\alpha_H = n \cdot P([0,1/n])$, so $P([0,1/n]) = \alpha_H \cdot (1/n)$.
- For $[0,m/n] = [0,1/n] + \ldots + [0,1/n]$ (m times), additivity implies $P([0,m/n]) = \alpha_H \cdot (m/n)$.
- For each real number r, for each n, there is an m s.t. $m/n \le r \le (m+1)/n$.
- Monotonicity implies $\alpha_H \cdot (m/n) = P([0,m/n]) \le P([0,r]) \le P([0,(m+1)/n]) = \alpha_H \cdot ((m+1)/n)$.
- When $n \to \infty$, $\alpha_H \cdot (m/n) \to \alpha_H \cdot r$ and $\alpha_H \cdot ((m+1)/n) \to \alpha_H \cdot r$, hence $P([0,r]) = \alpha_H \cdot r$.
- For $[\underline{u}, \overline{u}] = [\underline{u}, \underline{u}] + [0, \overline{u} - \underline{u}]$, additivity implies $P([\underline{u}, \overline{u}]) = \underline{u} + \alpha_H \cdot (\overline{u} - \underline{u})$. Q.E.D.

16. Case of Set-Valued Uncertainty

- In some cases:
  - in addition to knowing that the actual gain belongs to the interval $[\underline{u}, \overline{u}]$,
  - we also know that some values from this interval cannot be possible values of this gain.
- For example:
  - if we buy an obscure lottery ticket for a simple prize-or-no-prize lottery from a remote country,
  - we either get the prize or lose the money.
- In this case, the set of possible values of the gain consists of two values.
- Instead of a (bounded) *interval* of possible values, we can consider a general bounded *set* of possible values.

17. Fair Price Under Set-Valued Uncertainty

- We want a function P that assigns, to every bounded closed set S, a real number $P(S)$, for which:
  - $P([\underline{u}, \overline{u}]) = \alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u}$ (*conservativeness*);
  - $P(S + S') = P(S) + P(S')$, where $S + S' \stackrel{\text{def}}{=} \{s + s' : s \in S, s' \in S'\}$ (*additivity*).
- *Theorem:* Each fair price under set uncertainty has the form $P(S) = \alpha_H \cdot \sup S + (1 - \alpha_H) \cdot \inf S$.
- *Proof: idea.*
  - $\{\underline{s}, \overline{s}\} \subseteq S \subseteq [\underline{s}, \overline{s}]$, where $\underline{s} \stackrel{\text{def}}{=} \inf S$ and $\underline{s} \stackrel{\text{def}}{=} \sup S$;
  - thus, $[2\underline{s}, 2\overline{s}] = \{\underline{s}, \overline{s}\} + [\underline{s}, \overline{s}] \subseteq S + [\underline{s}, \overline{s}] \subseteq [\underline{s}, \overline{s}] + [\underline{s}, \overline{s}] = [2\underline{s}, 2\overline{s}]$;
  - so $S + [\underline{s}, \overline{s}] = [2\underline{s}, 2\overline{s}]$, hence $P(S) + P([\underline{s}, \overline{s}]) = P([2\underline{s}, 2\overline{s}])$, and

$$P(S) = (\alpha_H \cdot (2\overline{s}) + (1 - \alpha_H) \cdot (2\underline{s})) - (\alpha_H \cdot \overline{s} + (1 - \alpha_H) \cdot \underline{s}).$$

18. Case of Probabilistic Uncertainty

- Suppose that for some financial instrument, we know a prob. distribution $\rho(x)$ on the set of possible gains x.
- What is the fair price P for this instrument?
- Due to additivity, the fair price for n copies of this instrument is $n \cdot P$.
- According to the Large Numbers Theorem, for large n, the average gain tends to the mean value

$$\mu = \int x \cdot \rho(x)\, dx.$$

- Thus, the fair price for n copies of the instrument is close to $n \cdot \mu$: $n \cdot P \approx n \cdot \mu$.
- The larger n, the closer the averages. So, in the limit, we get $P = \mu$.

19. Case of p-Box Uncertainty

- Probabilistic uncertainty means that for every x, we know the value of the cdf $F(x) = \text{Prob}(\eta \le x)$.
- In practice, we often only have partial information about these values.
- In this case, for each x, we only know an interval $[\underline{F}(x), \overline{F}(x)]$ containing the actual (unknown) value $F(x)$.
- The interval-valued function $[\underline{F}(x), \overline{F}(x)]$ is known as a *p-box*.
- What is the fair price of a p-box?
- The only information that we have about the cdf is that $F(x) \in [\underline{F}(x), \overline{F}(x)]$.
- For each possible $F(x)$, for large n, n copies of the instrument are $\approx$ equivalent to $n \cdot \mu$, w/ $\mu = \int x\,dF(x)$.

20. Case of p-Box Uncertainty (cont-d)

- For each possible $F(x)$, for large n, n copies of the instrument are $\approx$ equivalent to $n \cdot \mu$, where

$$\mu = \int x\, dF(x).$$

- For different $F(x)$, values of μ for an interval $[\underline{\mu}, \overline{\mu}]$, where $\underline{\mu} = \int x\, d\overline{F}(x)$ and $\overline{\mu} = \int x\, d\underline{F}(x)$.
- Thus, the price of a p-box is equal to the price of an interval $[\underline{\mu}, \overline{\mu}]$.
- We already know that this price is equal to

$$\alpha_H \cdot \overline{\mu} + (1 - \alpha_H) \cdot \underline{\mu}.$$

- So, this is a fair price of a p-box.

21. Case of Twin Intervals

- Sometimes, in addition to the interval $[\underline{x}, \overline{x}]$, we also have a "most probable" subinterval $[\underline{m}, \overline{m}] \subseteq [\underline{x}, \overline{x}]$.
- For such "twin intervals", addition is defined component-wise:

$$([\underline{x}, \overline{x}], [\underline{m}, \overline{m}]) + ([\underline{y}, \overline{y}], [\underline{n}, \overline{n}]) = ([\underline{x}+\underline{y}, \overline{x}+\overline{y}], [\underline{m}+\underline{n}, \overline{m}+\overline{n}]).$$

- Thus, the additivity for additivity requirement about the fair prices takes the form

$$P([\underline{x}+\underline{y}, \overline{x}+\overline{y}], [\underline{m}+\underline{n}, \overline{m}+\overline{n}]) =$$

$$P([\underline{x}, \overline{x}], [\underline{m}, \overline{m}]) + P([\underline{y}, \overline{y}], [\underline{n}, \overline{n}]).$$

22. Fair Price Under Twin Interval Uncertainty

- By a *fair price under twin uncertainty*, we mean a function $P([\underline{u}, \overline{u}], [\underline{m}, \overline{m}])$ for which:
  - $\underline{u} \le P([\underline{u}, \overline{u}], [\underline{m}, \overline{m}]) \le \overline{u}$ for all $\underline{u} \le \underline{m} \le \overline{m} \le \overline{u}$ (*conservativeness*);
  - if $\underline{u} \le \underline{v}$, $\underline{m} \le \underline{n}$, $\overline{m} \le \overline{n}$, and $\overline{u} \le \overline{v}$, then $P([\underline{u}, \overline{u}], [\underline{m}, \overline{m}]) \le P([\underline{v}, \overline{v}], [\underline{n}, \overline{n}])$ (*monotonicity*);
  - for all $\underline{u} \le \underline{m} \le \overline{m} \le \overline{u}$ and $\underline{v} \le \underline{n} \le \overline{n} \le \overline{v}$, we have *additivity*:

$$P([\underline{u}+\underline{v}, \overline{u}+\overline{v}], [\underline{m}+\underline{n}, \overline{m}+\overline{m}]) = P([\underline{u}, \overline{u}], [\underline{m}, \overline{m}]) + P([\underline{v}, \overline{v}], [\underline{n}, \overline{n}]).$$

- *Theorem:* Each fair price under twin uncertainty has the following form, for some $\alpha_L, \alpha_u, \alpha_U \in [0, 1]$:

$$P([\underline{u}, \overline{u}], [\underline{m}, \overline{m}]) = \underline{m} + \alpha_u \cdot (\overline{m} - \underline{m}) + \alpha_U \cdot (\overline{U} - \overline{m}) + \alpha_L \cdot (\underline{u} - \underline{m}).$$

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23. Case of Fuzzy Numbers

- An expert is often imprecise ("fuzzy") about the possible values.
- For example, an expert may say that the gain is small.
- To describe such information, L. Zadeh introduced the notion of *fuzzy numbers*.
- For fuzzy numbers, different values u are possible with different degrees $\mu(u) \in [0, 1]$.
- The value w is a possible value of $u + v$ if:
  - for some values u and v for which $u + v = w$,
  - u is a possible value of 1st gain, and
  - v is a possible value of 2nd gain.
- If we interpret "and" as min and "or" ("for some") as max, we get *Zadeh's extension principle:*

$$\mu(w) = \max_{u,v:\, u+v=w} \min(\mu_1(u), \mu_2(v)).$$

24. Case of Fuzzy Numbers (cont-d)

- *Reminder:* $\mu(w) = \max\limits_{u,v:\, u+v=w} \min(\mu_1(u), \mu_2(v))$.
- This operation is easiest to describe in terms of α-cuts

$$\mathbf{u}(\alpha) = [u^-(\alpha), u^+(\alpha)] \stackrel{\text{def}}{=} \{u : \mu(u) \ge \alpha\}.$$

- Namely, $\mathbf{w}(\alpha) = \mathbf{u}(\alpha) + \mathbf{v}(\alpha)$, i.e.,

$$w^-(\alpha) = u^-(\alpha) + v^-(\alpha) \text{ and } w^+(\alpha) = u^+(\alpha) + v^+(\alpha).$$

- For product (of probabilities), we similarly get

$$\mu(w) = \max_{u,v:\, u\cdot v=w} \min(\mu_1(u), \mu_2(v)).$$

- In terms of α-cuts, we have $\mathbf{w}(\alpha) = \mathbf{u}(\alpha) \cdot \mathbf{v}(\alpha)$, i.e.,

$$w^-(\alpha) = u^-(\alpha) \cdot v^-(\alpha) \text{ and } w^+(\alpha) = u^+(\alpha) \cdot v^+(\alpha).$$

25. Fair Price Under Fuzzy Uncertainty

- We want to assign, to every fuzzy number s, a real number $P(s)$, so that:
  - if a fuzzy number s is located between $\underline{u}$ and $\overline{u}$, then $\underline{u} \le P(s) \le \overline{u}$ (*conservativeness*);
  - $P(u+v) = P(u) + P(v)$ (*additivity*);
  - if for all α, $s^-(\alpha) \le t^-(\alpha)$ and $s^+(\alpha) \le t^+(\alpha)$, then we have $P(s) \le P(t)$ (*monotonicity*);
  - if μ_n uniformly converges to μ, then $P(\mu_n) \to P(\mu)$ (*continuity*).
- *Theorem.* The fair price is equal to

$$P(s) = s_0 + \int_0^1 k^-(\alpha)\,ds^-(\alpha) - \int_0^1 k^+(\alpha)\,ds^+(\alpha) \text{ for some } k^\pm(\alpha).$$

26. Discussion

- $\int f(x) \cdot dg(x) = \int f(x) \cdot g'(x)\, dx$ for a *generalized function* $g'(x)$, hence for generalized $K^{\pm}(\alpha)$, we have:
$$P(s) = \int_0^1 K^-(\alpha) \cdot s^-(\alpha)\, d\alpha + \int_0^1 K^+(\alpha) \cdot s^+(\alpha)\, d\alpha.$$
- Conservativeness means that
$$\int_0^1 K^-(\alpha)\, d\alpha + \int_0^1 K^+(\alpha)\, d\alpha = 1.$$
- For the interval $[\underline{u}, \overline{u}]$, we get
$$P(s) = \left(\int_0^1 K^-(\alpha)\, d\alpha\right) \cdot \underline{u} + \left(\int_0^1 K^+(\alpha)\, d\alpha\right) \cdot \overline{u}.$$
- Thus, Hurwicz optimism-pessimism coefficient α_H is equal to $\int_0^1 K^+(\alpha)\, d\alpha$.
- In this sense, the above formula is a generalization of Hurwicz's formula to the fuzzy case.

Part III
Utility-Based Approach

27. Monetary Approach Is Not Always Appropriate

- In some situations, the result of the decision is the decision maker's own satisfaction.
- Examples:
  - buying a house to live in,
  - selecting a movie to watch.
- In such situations, monetary approach is not appropriate.
- For example:
  - a small apartment in downtown can be very expensive,
  - but many people prefer a cheaper – but more spacious and comfortable – suburban house.

28. Non-Monetary Decision Making: Traditional Approach

- To make a decision, we must:
  - find out the user's preference, and
  - help the user select an alternative which is the best – according to these preferences.
- Traditional approach is based on an assumption that for each two alternatives A' and A'', a user can tell:
  - whether the first alternative is better for him/her; we will denote this by $A'' < A'$;
  - or the second alternative is better; we will denote this by $A' < A''$;
  - or the two given alternatives are of equal value to the user; we will denote this by $A' = A''$.

29. The Notion of Utility

- Under the above assumption, we can form a natural numerical scale for describing preferences.
- Let us select a very bad alternative A_0 and a very good alternative A_1.
- Then, most other alternatives are better than A_0 but worse than A_1.
- For every prob. $p \in [0, 1]$, we can form a lottery $L(p)$ in which we get A_1 w/prob. p and A_0 w/prob. $1 - p$.
- When $p = 0$, this lottery simply coincides with the alternative A_0: $L(0) = A_0$.
- The larger the probability p of the positive outcome increases, the better the result:

$$p' < p'' \text{ implies } L(p') < L(p'').$$

30. The Notion of Utility (cont-d)

- Finally, for $p = 1$, the lottery coincides with the alternative A_1: $L(1) = A_1$.
- Thus, we have a continuous scale of alternatives $L(p)$ that monotonically goes from $L(0) = A_0$ to $L(1) = A_1$.
- Due to monotonicity, when p increases, we first have $L(p) < A$, then we have $L(p) > A$.
- The threshold value is called the *utility* of the alternative A:

$$u(A) \stackrel{\text{def}}{=} \sup\{p : L(p) < A\} = \inf\{p : L(p) > A\}.$$

- Then, for every $\varepsilon > 0$, we have

$$L(u(A) - \varepsilon) < A < L(u(A) + \varepsilon).$$

- We will describe such (almost) equivalence by $\equiv$, i.e., we will write that $A \equiv L(u(A))$.

31. Fast Iterative Process for Determining $u(A)$

- *Initially:* we know the values $\underline{u} = 0$ and $\overline{u} = 1$ such that $A \equiv L(u(A))$ for some $u(A) \in [\underline{u}, \overline{u}]$.
- *What we do:* we compute the midpoint $u_{\rm mid}$ of the interval $[\underline{u}, \overline{u}]$ and compare A with $L(u_{\rm mid})$.
- *Possibilities:* $A \le L(u_{\rm mid})$ and $L(u_{\rm mid}) \le A$.
- *Case 1:* if $A \le L(u_{\rm mid})$, then $u(A) \le u_{\rm mid}$, so
$$u \in [\underline{u}, u_{\rm mid}].$$
- *Case 2:* if $L(u_{\rm mid}) \le A$, then $u_{\rm mid} \le u(A)$, so
$$u \in [u_{\rm mid}, \overline{u}].$$
- After each iteration, we decrease the width of the interval $[\underline{u}, \overline{u}]$ by half.
- After k iterations, we get an interval of width 2^{-k} which contains $u(A)$ – i.e., we get $u(A)$ w/accuracy 2^{-k}.

32. How to Make a Decision Based on Utility Values

- Suppose that we have found the utilities $u(A')$, $u(A'')$, …, of the alternatives A', A'', …
- Which of these alternatives should we choose?
- By definition of utility, we have:
  - $A \equiv L(u(A))$ for every alternative A, and
  - $L(p') < L(p'')$ if and only if $p' < p''$.
- We can thus conclude that A' is preferable to A'' if and only if $u(A') > u(A'')$.
- In other words, we should always select an alternative with the largest possible value of utility.
- Interval techniques can help in finding the optimizing decision.

33. How to Estimate Utility of an Action

- For each action, we usually know possible outcomes $S_1, \ldots, S_n$.
- We can often estimate the prob. $p_1, \ldots, p_n$ of these outcomes.
- By definition of utility, each situation S_i is equiv. to a lottery $L(u(S_i))$ in which we get:
  - A_1 with probability $u(S_i)$ and
  - A_0 with the remaining probability $1 - u(S_i)$.
- Thus, the action is equivalent to a complex lottery in which:
  - first, we select one of the situations S_i with probability p_i: $P(S_i) = p_i$;
  - then, depending on S_i, we get A_1 with probability $P(A_1 \mid S_i) = u(S_i)$ and A_0 w/probability $1 - u(S_i)$.

34. How to Estimate Utility of an Action (cont-d)

- *Reminder:*
  - first, we select one of the situations S_i with probability p_i: $P(S_i) = p_i$;
  - then, depending on S_i, we get A_1 with probability $P(A_1 \mid S_i) = u(S_i)$ and A_0 w/probability $1 - u(S_i)$.
- The prob. of getting A_1 in this complex lottery is:

$$P(A_1) = \sum_{i=1}^{n} P(A_1 \mid S_i) \cdot P(S_i) = \sum_{i=1}^{n} u(S_i) \cdot p_i.$$

- In the complex lottery, we get:
  - A_1 with prob. $u = \sum\limits_{i=1}^{n} p_i \cdot u(S_i)$, and
  - A_0 w/prob. $1 - u$.
- So, we should select the action with the largest value of expected utility $u = \sum p_i \cdot u(S_i)$.

35. Utility Is Different from Money

- Empirical data shows that utility u is proportional to square root of money x:
  - when $x > 0$, we have $u(x) = c_+ \cdot \sqrt{x}$;
  - when $x < 0$, we have $u(x) = -c_- \cdot \sqrt{|x|}$.
- This explains why most people are risk-averse.
- Indeed, let us consider two cases:
  - getting \$50, and
  - getting \$100 with probability 0.5.
- In both cases, the expected amount is the same, but:
  - in the first case, $u(x) = c_+ \cdot \sqrt{50} \approx 7 \cdot c_+$;
  - in the second case, the expected utility is

$$0.5 \cdot c_+ \cdot \sqrt{100} + 0.5 \cdot c_+ \cdot \sqrt{0} = 5 \cdot c_+ \ll 7 \cdot c_+.$$

36. Non-Uniqueness of Utility

- The above definition of utility u depends on A_0, A_1.
- What if we use different alternatives A'_0 and A'_1?
- Every A is equivalent to a lottery $L(u(A))$ in which we get A_1 w/prob. $u(A)$ and A_0 w/prob. $1 - u(A)$.
- For simplicity, let us assume that $A'_0 < A_0 < A_1 < A'_1$.
- Then, $A_0 \equiv L'(u'(A_0))$ and $A_1 \equiv L'(u'(A_1))$.
- So, A is equivalent to a complex lottery in which:
  1) we select A_1 w/prob. $u(A)$ and A_0 w/prob. $1-u(A)$;
  2) depending on A_i, we get A'_1 w/prob. $u'(A_i)$ and A'_0 w/prob. $1 - u'(A_i)$.
- In this complex lottery, we get A'_1 with probability $u'(A) = u(A) \cdot (u'(A_1) - u'(A_0)) + u'(A_0)$.
- So, in general, utility is defined modulo an (increasing) linear transformation $u' = a \cdot u + b$, with $a > 0$.

37. Subjective Probabilities

- In practice, we often do not know the probabilities p_i of different outcomes.
- For each event E, a natural way to estimate its subjective probability is to fix a prize (e.g., \$1) and compare:
  - the lottery ℓ_E in which we get the fixed prize if the event E occurs and 0 is it does not occur, with
  - a lottery $\ell(p)$ in which we get the same amount with probability p.
- Here, similarly to the utility case, we get a value $ps(E)$ for which, for every $\varepsilon > 0$:

$$\ell(ps(E) - \varepsilon) < \ell_E < \ell(ps(E) + \varepsilon).$$

- Then, the utility of an action with possible outcomes $S_1, \ldots, S_n$ is equal to $u = \sum_{i=1}^{n} ps(E_i) \cdot u(S_i)$.

38. Beyond Traditional Decision Making: Towards a More Realistic Description

- Previously, we assumed that a user can always decide which of the two alternatives A' and A'' is better:
  - either $A' < A''$,
  - or $A'' < A'$,
  - or $A' \equiv A''$.
- In practice, a user is sometimes unable to meaningfully decide between the two alternatives; denoted $A' \parallel A''$.
- In mathematical terms, this means that the preference relation:
  - is no longer a *total* (linear) order,
  - it can be a *partial* order.

39. From Utility to Interval-Valued Utility

- Similarly to the traditional decision making approach:
  - we select two alternatives $A_0 < A_1$ and
  - we compare each alternative A which is better than A_0 and worse than A_1 with lotteries $L(p)$.
- Since preference is a *partial* order, in general:

$$\underline{u}(A) \stackrel{\text{def}}{=} \sup\{p : L(p) < A\} < \overline{u}(A) \stackrel{\text{def}}{=} \inf\{p : L(p) > A\}.$$

- For each alternative A, instead of a single value $u(A)$ of the utility, we now have an *interval* $[\underline{u}(A), \overline{u}(A)]$ s.t.:
  - if $p < \underline{u}(A)$, then $L(p) < A$;
  - if $p > \overline{u}(A)$, then $A < L(p)$; and
  - if $\underline{u}(A) < p < \overline{u}(A)$, then $A \parallel L(p)$.
- We will call this interval the *utility* of the alternative A.

40. Interval-Valued Utilities and Interval-Valued Subjective Probabilities

- To feasibly elicit the values $\underline{u}(A)$ and $\overline{u}(A)$, we:
  1) starting w/$[\underline{u}, \overline{u}] = [0, 1]$, bisect an interval s.t. $L(\underline{u}) < A < L(\overline{u})$ until we find u_0 s.t. $A \parallel L(u_0)$;
  2) by bisecting an interval $[\underline{u}, u_0]$ for which $L(\underline{u}) < A \parallel L(u_0)$, we find $\underline{u}(A)$;
  3) by bisecting an interval $[u_0, \overline{u}]$ for which $L(u_0) \parallel A < L(\overline{u})$, we find $\overline{u}(A)$.
- Similarly, when we estimate the probability of an event E:
  - we no longer get a single value $ps(E)$;
  - we get an *interval* $\left[\underline{ps}(E), \overline{ps}(E)\right]$ of possible values of probability.
- By using bisection, we can feasibly elicit the values $\underline{ps}(E)$ and $\overline{ps}(E)$.

41. Decision Making Under Interval Uncertainty

- *Situation:* for each possible decision d, we know the interval $[\underline{u}(d), \overline{u}(d)]$ of possible values of utility.
- *Questions:* which decision shall we select?
- *Natural idea:* select all decisions d_0 that *may* be optimal, i.e., which are optimal for some function

$$u(d) \in [\underline{u}(d), \overline{u}(d)].$$

- *Problem:* checking all possible functions is not feasible.
- *Solution:* the above condition is equivalent to an easier-to-check one:

$$\overline{u}(d_0) \geq \max_d \underline{u}(d).$$

- *Interval computations* can help in describing the range of all such d_0.
- *Remaining problem:* in practice, we would like to select *one* decision; which one should be select?

42. Need for Definite Decision Making

- *At first glance:* if $A' \parallel A''$, it does not matter whether we recommend alternative A' or alternative A''.
- Let us show that this is *not* a good recommendation.
- E.g., let A be an alternative about which we know nothing, i.e., $[\underline{u}(A), \overline{u}(A)] = [0, 1]$.
- In this case, A is indistinguishable both from a "good" lottery $L(0.999)$ and a "bad" lottery $L(0.001)$.
- Suppose that we recommend, to the user, that A is equivalent both to $L(0.999)$ and to $L(0.001)$.
- Then this user will feel comfortable:
  - first, exchanging $L(0.999)$ with A, and
  - then, exchanging A with $L(0.001)$.
- So, following our recommendations, the user switches from a very good alternative to a very bad one.

43. Need for Definite Decision Making (cont-d)

- The above argument does not depend on the fact that we assumed complete ignorance about A:
  - every time we recommend that the alternative A is "equivalent" both to $L(p)$ and to $L(p')$ $(p < p')$,
  - we make the user vulnerable to a similar switch from a better alternative $L(p')$ to a worse one $L(p)$.
- Thus, there should be only a single value p for which A can be reasonably exchanged with $L(p)$.
- In precise terms:
  - we start with the utility interval $[\underline{u}(A), \overline{u}(A)]$, and
  - we need to select a single $u(A)$ for which it is reasonable to exchange A with a lottery $L(u)$.
- How can we find this value $u(A)$?

44. Decisions under Interval Uncertainty: Hurwicz Optimism-Pessimism Criterion

- *Reminder:* we need to assign, to each interval $[\underline{u}, \overline{u}]$, a utility value $u(\underline{u}, \overline{u}) \in [\underline{u}, \overline{u}]$.
- *History:* this problem was first handled in 1951, by the future Nobelist Leonid Hurwicz.
- *Notation:* let us denote $\alpha_H \stackrel{\text{def}}{=} u(0, 1)$.
- *Reminder:* utility is determined modulo a linear transformation $u' = a \cdot u + b$.
- *Reasonable to require:* the equivalent utility does not change with re-scaling: for $a > 0$ and b,

$$u(a \cdot u^- + b, a \cdot u^+ + b) = a \cdot u(u^-, u^+) + b.$$

- For $u^- = 0$, $u^+ = 1$, $a = \overline{u} - \underline{u}$, and $b = \underline{u}$, we get

$$u(\underline{u}, \overline{u}) = \alpha_H \cdot (\overline{u} - \underline{u}) + \underline{u} = \alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u}.$$

45. Hurwicz Optimism-Pessimism Criterion (cont)

- The expression $\alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u}$ is called *optimism-pessimism criterion*, because:
  - when $\alpha_H = 1$, we make a decision based on the most optimistic possible values $u = \overline{u}$;
  - when $\alpha_H = 0$, we make a decision based on the most pessimistic possible values $u = \underline{u}$;
  - for intermediate values $\alpha_H \in (0, 1)$, we take a weighted average of the optimistic and pessimistic values.
- According to this criterion:
  - if we have several alternatives A', ..., with interval-valued utilities $[\underline{u}(A'), \overline{u}(A')]$, ...,
  - we recommend an alternative A that maximizes

$$\alpha_H \cdot \overline{u}(A) + (1 - \alpha_H) \cdot \underline{u}(A).$$

46. Which Value α_H Should We Choose? An Argument in Favor of $\alpha_H = 0.5$

- Let us take an event E about which we know nothing.
- For a lottery L^+ in which we get A_1 if E and A_0 otherwise, the utility interval is $[0, 1]$.
- Thus, the equiv. utility of L^+ is $\alpha_H \cdot 1 + (1 - \alpha_H) \cdot 0 = \alpha_H$.
- For a lottery L^- in which we get A_0 if E and A_1 otherwise, the utility is $[0, 1]$, so equiv. utility is also α_H.
- For a complex lottery L in which we select either L^+ or L^- with probability 0.5, the equiv. utility is still α_H.
- On the other hand, in L, we get A_1 with probability 0.5 and A_0 with probability 0.5.
- Thus, $L = L(0.5)$ and hence, $u(L) = 0.5$.
- So, we conclude that $\alpha_H = 0.5$.

47. Which Action Should We Choose?

- Suppose that an action has n possible outcomes $S_1, \ldots, S_n$, with utilities $[\underline{u}(S_i), \overline{u}(S_i)]$, and probabilities $[\underline{p}_i, \overline{p}_i]$.
- We know that each alternative is equivalent to a simple lottery with utility $u_i = \alpha_H \cdot \overline{u}(S_i) + (1 - \alpha_H) \cdot \underline{u}(S_i)$.
- We know that for each i, the i-th event is equivalent to $p_i = \alpha_H \cdot \overline{p}_i + (1 - \alpha_H) \cdot \underline{p}_i$.
- Thus, this action is equivalent to a situation in which we get utility u_i with probability p_i.
- The utility of such a situation is equal to $\sum\limits_{i=1}^{n} p_i \cdot u_i$.
- Thus, the equivalent utility of the original action is equivalent to

$$\sum_{i=1}^{n} \left(\alpha_H \cdot \overline{p}_i + (1 - \alpha_H) \cdot \underline{p}_i\right) \cdot (\alpha_H \cdot \overline{u}(S_i) + (1 - \alpha_H) \cdot \underline{u}(S_i)) \, .$$

48. Observation: the Resulting Decision Depends on the Level of Detail

- Let us consider a situation in which, with some prob. p, we gain a utility u, else we get 0.
- The expected utility is $p \cdot u + (1 - p) \cdot 0 = p \cdot u$.
- Suppose that we only know the intervals $[\underline{u}, \overline{u}]$ and $[\underline{p}, \overline{p}]$.
- The equivalent utility u_k (k for *k*now) is

$$u_k = (\alpha_H \cdot \overline{p} + (1 - \alpha_H) \cdot \underline{p}) \cdot (\alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u}).$$

- If we only know that utility is from $[\underline{p} \cdot \underline{u}, \overline{p} \cdot \overline{u}]$, then:

$$u_d = \alpha_H \cdot \overline{p} \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{p} \cdot \underline{u} \ (d \text{ for } d\text{on't know}).$$

- Here, additional knowledge decreases utility:

$$u_d - u_k = \alpha_H \cdot (1 - \alpha_H) \cdot (\overline{p} - \underline{p}) \cdot (\overline{u} - \underline{u}) > 0.$$

- (This is maybe what the Book of Ecclesiastes meant by "For with much wisdom comes much sorrow"?)

49. Beyond Interval Uncertainty: Partial Info about Probabilities

- *Frequent situation:*
  - in addition to $\mathbf{x}_i$,
  - we may also have *partial* information about the probabilities of different values $x_i \in \mathbf{x}_i$.
- An *exact* probability distribution can be described, e.g., by its cumulative distribution function

$$F_i(z) = \text{Prob}(x_i \le z).$$

- A *partial* information means that instead of a single cdf, we have a *class* $\mathcal{F}$ of possible cdfs.
- *p-box* (Scott Ferson):
  - for every z, we know an interval $\mathbf{F}(z) = [\underline{F}(z), \overline{F}(z)]$;
  - we consider all possible distributions for which, for all z, we have $F(z) \in \mathbf{F}(z)$.

50. Describing Partial Info about Probabilities: Decision Making Viewpoint

- *Problem:* there are many ways to represent a probability distribution.
- *Idea:* look for an objective.
- *Objective:* make decisions $E_x[u(x,a)] \to \max\limits_a$.
- *Case 1:* smooth $u(x)$.
- *Analysis:* we have $u(x) = u(x_0) + (x - x_0) \cdot u'(x_0) + \ldots$
- *Conclusion:* we must know moments to estimate $E[u]$.
- *Case of uncertainty:* interval bounds on moments.
- *Case 2:* threshold-type $u(x)$ (e.g., regulations).
- *Conclusion:* we need cdf $F(x) = \text{Prob}(\xi \le x)$.
- *Case of uncertainty:* p-box $[\underline{F}(x), \overline{F}(x)]$.

Part IV
Group Decision Making

51. Multi-Agent Cooperative Decision Making

- *How to describe preferences:* for each participant P_i, we can determine the utility $u_{ij} \stackrel{\text{def}}{=} u_i(A_j)$ of all A_j.
- *Question:* how to transform these utilities into a reasonable group decision rule?
- *Solution:* was provided by another future Nobelist John Nash.
- *Nash's assumptions:*
  - symmetry,
  - independence from irrelevant alternatives, and
  - *scale invariance* – under replacing function $u_i(A)$ with an equivalent function $a \cdot u_i(A)$,

52. Nash's Bargaining Solution (cont-d)

- *Nash's assumptions (reminder):*
  - symmetry,
  - independence from irrelevant alternatives, and
  - scale invariance.
- *Nash's result:*
  - the only group decision rule satisfying all these assumptions
  - is selecting an alternative A for which the product $\prod\limits_{i=1}^{n} u_i(A)$ is the largest possible.
- *Comment.* the utility functions must be "scaled" s.t. the "status quo" situation $A^{(0)}$ has utility 0:

$$u_i(A) \to u_i'(A) \stackrel{\text{def}}{=} u_i(A) - u_i(A^{(0)}).$$

53. Multi-Agent Decision Making under Interval Uncertainty

- *Reminder:* if we set utility of status quo to 0, then we select an alternative A that maximizes
$$u(A) = \prod_{i=1}^{n} u_i(A).$$
- *Case of interval uncertainty:* we only know intervals $[\underline{u}_i(A), \overline{u}_i(A)]$.
- *First idea:* find all A_0 for which $\overline{u}(A_0) \geq \max_A \underline{u}(A)$, where
$$[\underline{u}(A), \overline{u}(A)] \stackrel{\text{def}}{=} \prod_{i=1}^{n} [\underline{u}_i(A), \overline{u}_i(A)].$$
- *Second idea:* maximize $u^{\text{equiv}}(A) \stackrel{\text{def}}{=} \prod_{i=1}^{n} u_i^{\text{equiv}}(A)$.
- *Interesting aspect:* when we have a conflict situation (e.g., in security games).

54. Group Decision Making and Arrow's Impossibility Theorem

- In 1951, Kenneth J. Arrow published his famous result about group decision making.
- This result that became one of the main reasons for his 1972 Nobel Prize.
- *The problem:*
  - A group of n participants $P_1, \ldots, P_n$ needs to select between one of m alternatives $A_1, \ldots, A_m$.
  - To find individual preferences, we ask each participant P_i to rank the alternatives A_j:

$$A_{j_1} \succ_i A_{j_2} \succ_i \ldots \succ_i A_{j_n}.$$

  - Based on these n rankings, we must form a single group ranking (equivalence $\sim$ is allowed).

55. Case of Two Alternatives Is Easy

- *Simplest case:*
  - we have only two alternatives A_1 and A_2,
  - each participant either prefers A_1 or prefers A_2.
- *Solution:* it is reasonable, for a group:
  - to prefer A_1 if the majority prefers A_1,
  - to prefer A_2 if the majority prefers A_2, and
  - to claim A_1 and A_2 to be of equal quality for the group (denoted $A_1 \sim A_2$) if there is a tie.

56. Case of Three or More Alternatives Is Not Easy

- *Arrow's result:* no group decision rule can satisfy the following natural conditions.
- *Pareto condition:* if all participants prefer A_j to A_k, then the group should also prefer A_j to A_k.
- *Independence from Irrelevant Alternatives:* the group ranking of A_j vs. A_k should not depend on other A_is.
- *Arrow's theorem:* every group decision rule which satisfies these two condition is a *dictatorship* rule:
  - the group accepts the preferences of one of the participants as the group decision and
  - ignores the preferences of all other participants.
- This violates *symmetry*: that the group decision rules should not depend on the order of the participants.

57. Beyond Arrow's Impossibility Theorem

- *Usual claim:* Arrow's Impossibility Theorem proves that reasonable group decision making is impossible.
- *Our claim:* Arrow's result is only valid if we have binary ("yes"-"no") individual preferences.
- *Fact:* this information does not fully describe a persons' preferences.
- *Example:* the preference $A_1 \succ A_2 \succ A_3$:
  - it may indicate that a person strongly prefers A_1 to A_2, and strongly prefers A_2 to A_3, and
  - it may also indicate that this person strongly prefers A_1 to A_2, and at the same time, $A_2 \approx A_3$.
- *How can this distinction be described:* researchers in decision making use the notion of *utility*.

58. Nash's Solution as a Way to Overcome Arrow's Paradox

- *Situation:* for each participant P_i $(i = 1, \ldots, n)$, we know his/her utility $u_i(A_j)$ of A_j, $j = 1, \ldots, m$.
- *Possible choices:* lotteries $p = (p_1, \ldots, p_m)$ in which we select A_j with probability $p_j \geq 0$, $\sum\limits_{j=1}^{m} p_j = 1$.
- *Nash's solution:* among all the lotteries p, we select the one that maximizes

$$\prod_{i=1}^{n} u_i(p), \text{ where } u_i(p) = \sum_{j=1}^{m} p_j \cdot u_i(A_j).$$

- *Generic case:* no two vectors $u_i = (u_i(A_1), \ldots, u_i(A_m))$ are collinear.
- *In this general case:* Nash's solution is unique.

59. Sometimes It Is Beneficial to Cheat: An Example

- *Situation:* participant P_1 know the utilities of all the other participants, but they don't know his $u_1(B)$.
- *Notation:* let A_{m_1} be P_1's best alternative:

$$u_1(A_{m_1}) \geq u_1(A_j) \text{ for all } j \neq m_1.$$

- *How to cheat:* P_1 can force the group to select A_{m_1} by using a "fake" utility function $u'_1(A)$ for which
  - $u'_1(A_{m_1}) = 1$ and
  - $u'_1(A_j) = 0$ for all $j \neq m_1$.
- *Why it works:*
  - when selecting A_j w/$j \neq m_1$, we get $\prod u_i(A_j) = 0$;
  - when selecting A_{m_1}, we get $\prod u_i(A_j) > 0$.

60. Cheating May Hurt the Cheater: an Observation

- *A more typical situation:* no one knows others' utility functions.
- Let P_1 use the above false utility function $u'_1(A)$ for which $u'_1(A_{m_1}) = 1$ and $u'_1(A_j) = 0$ for all $j \neq m_1$.
- *Possibility:* others use similar utilities with $u_i(A_{m_i}) > 0$ for some $m_i \neq m_1$ and $u_i(A_j) = 0$ for $j \neq m_i$.
- Then for every alternative A_j, Nash's product is equal to 0.
- From this viewpoint, all alternatives are equally good, so each of them can be chosen.
- In particular, it may be possible that the group selects an alternative A_q which is *the worst* for P_1 – i.e., s.t.

$$u_1(A_q) < u_1(A_j) \text{ for all } j \neq p.$$

61. Case Study: Territorial Division

- Dividing a set (territory) A between n participants, i.e., finding X_i s.t. $\bigcup\limits_{i=1}^{n} X_i$ and $X_i \cap X_j = \emptyset$ for $i \neq j$.
- The utility functions have the form $u_i(X) = \int_X v_i(t)\,dt$.
- *Nash's solution:* maximize $u_1(X) \cdot \ldots \cdot u_n(X_n)$.
- *Assumption:* P_1 does not know $u_i(B)$ for $i \neq 1$.
- *Choices:* the participant P_1 can report a fake utility function $v_1'(t) \neq v_1(t)$.
- For each $v_1'(t)$, we maximizes the product

$$\left(\int_{X_1} v_1'(t)\,dt\right) \cdot \left(\int_{X_2} v_2(t)\,dt\right) \cdot \ldots \cdot \left(\int_{X_n} v_n(t)\,dt\right).$$

- *Question:* select $v_1'(t)$ that maximizes the gain

$$u(v_1', v_1, v_2, \ldots, v_n) \stackrel{\text{def}}{=} \int_{X_1} v_1'(t)\,dt.$$

62. For Territorial Division, It Is Beneficial to Report the Correct Utilities: Result

- *Hurwicz's criterion* $u(A) = \alpha \cdot u^-(A) + (1-\alpha) \cdot u^+(A)$ may sound arbitrary.
- *For our problem:* Hurwicz's criterion means that we select a utility function $v_1'(t)$ that maximizes

$$J(v_1') \stackrel{\text{def}}{=} \alpha \cdot \min_{v_2,\ldots,v_n} u(v_1', v_1, v_2, \ldots, v_n) +$$

$$(1-\alpha) \cdot \max_{v_2,\ldots,v_n} u(v_1', v_1, v_2, \ldots, v_n).$$

- *Theorem:* when $\alpha > 0$, the objective function $J(v_1')$ attains its largest possible value for $v_1'(t) = v_1(t)$.
- *Conclusion:* unless we select pure optimism, it is best to select $v_1'(t) = v_1(t)$, i.e., to tell the truth.

63. How to Find Individual Preferences from Collective Decision Making: Inverse Problem of Game Theory

- *Situation:* we have a group of n participants $P_1, \ldots, P_n$ that does not want to reveal its individual preferences.
- *Example:* political groups tend to hide internal disagreements.
- *Objective:* detect individual preferences.
- *Example:* this is waht kremlinologies used to do.
- *Assumption:* the group uses Nash's solution to make decisions.
- *We can:* ask the group as a whole to compare different alternatives.

64. Comment

- *Fact:* Nash's solution depends only on the product of the utility functions.
- *Corollary:* in the best case,
  - we will be able to determine n individual utility functions
  - without knowing which of these functions corresponds to which individual.
- *Comment:* this is OK, because
  - our main objective is to predict future behavior of this group,
  - and in this prediction, it is irrelevant who has which utility function.

65. How to Find Individual Preferences from Collective Decision Making: Our Result

- Let $u_{ij} = u_i(A_j)$ denote i-th utility of j-th alternative.
- *We assume* that utility is normalized: $u_i(A_0) = 0$ for status quo A_0 and $u_i(A_1) = 1$ for some A_1.
- *According to Nash:* $p = (p_1, \ldots, p_n) \preceq q = (q_1, \ldots, q_n) \Leftrightarrow$

$$\prod_{i=1}^{n} \left(\sum_{j=1}^{n} p_j \cdot u_{ij} \right) \le \prod_{i=1}^{n} \left(\sum_{j=1}^{n} q_j \cdot u_{ij} \right).$$

- *Theorem:* if utilities u_{ij} and u'_{ij} lead to the same preference $\preceq$, then they differ only by permutation.
- *Conclusion:* we can determine individual preferences from group decisions.
- *An efficient algorithm* for determining u_{ij} from $\preceq$ is possible.

66. We Must Take Altruism and Love into Account

- *Implicit assumption:* the utility $u_i(A_j)$ of a participant P_i depends only on what he/she gains.
- *In real life:* the degree of a person's happiness also depends on the degree of happiness of other people:
  - Normally, this dependence is positive, i.e., we feel happier if other people are happy.
  - However, negative emotions such as jealousy are also common.
- This idea was developed by another future Nobelist Gary Becker: $u_i = u_i^{(0)} + \sum\limits_{j \neq i} \alpha_{ij} \cdot u_j$, where:
  - $u_i^{(0)}$ is the utility of person i that does not take interdependence into account; and
  - u_j are utilities of other people $j \neq i$.

67. Paradox of Love

- *Case* $n = 2$*:* $u_1 = u_1^{(0)} + \alpha_{12} \cdot u_2$; $u_2 = u_2^{(0)} + \alpha_{21} \cdot u_1$.
- *Solution:* $u_1 = \dfrac{u_1^{(0)} + \alpha_{12} \cdot u_2^{(0)}}{1 - \alpha_{12} \cdot \alpha_{21}}$; $u_2 = \dfrac{u_2^{(0)} + \alpha_{21} \cdot u_1^{(0)}}{1 - \alpha_{12} \cdot \alpha_{21}}$.
- *Example:* mutual affection means that $\alpha_{12} > 0$ and $\alpha_{21} > 0$.
- *Example:* selfless love, when someone else's happiness means more than one's own, corresponds to $\alpha_{12} > 1$.
- *Paradox:*
  - when two people are deeply in love with each other ($\alpha_{12} > 1$ and $\alpha_{21} > 1$), then
  - positive original pleasures $u_i^{(0)} > 0$ lead to $u_i < 0$ – i.e., to unhappiness.

68. Paradox of Love: Discussion

- *Paradox – reminder:*
  - when two people are deeply in love with each other, then
  - positive original pleasures $u_i^{(0)} > 0$ lead to unhappiness.
- This may explain why people in love often experience deep negative emotions.
- From this viewpoint, a situation when
  - one person loves deeply and
  - another rather allows him- or herself to be loved

  may lead to more happiness than mutual passionate love.

69. Why Two and not Three?

- An *ideal love* is when each person treats other's emotions almost the same way as one's own, i.e.,

$$\alpha_{12} = \alpha_{21} = \alpha = 1 - \varepsilon \text{ for a small } \varepsilon > 0.$$

- For *two people*, from $u_i^{(0)} > 0$, we get $u_i > 0$ – i.e., we can still have happiness.
- For $n \geq 3$, even for $u_i^{(0)} = u^{(0)} > 0$, we get
$$u_i = \frac{u^{(0)}}{1 - (1 - \varepsilon) \cdot (n - 1)} < 0, \text{ i.e., unhappiness.}$$
- *Corollary:* if two people care about the same person (e.g., his mother and his wife),
  - all three of them are happier
  - if there is some negative feeling (e.g., jealousy) between them.

70. Emotional vs. Objective Interdependence

- *We considered:* emotional interdependence, when one's utility is determined by the utility of other people:

$$u_i = u_i^{(0)} + \sum_j \alpha_j \cdot u_j.$$

- *Alternative:* "objective" altruism, when one's utility depends on the material gain of other people:

$$u_i = u_i^{(0)} + \sum_j \alpha_j \cdot u_j^{(0)}.$$

- *In this approach:* we care about others' well-being, not about their emotions.
- *In this approach:* no paradoxes arise, any degree of altruism only improves the situation.
- The objective approach was proposed by yet another Nobel Prize winner Amartya K. Sen.

Part V
Case Study: Selecting a Location for a Meteorological Tower

71. Introduction

- *Challenge:* in many remote areas, meteorological sensor coverage is sparse.
- *Desirable:* design sensor networks that provide the largest amount of useful information within a given budget.
- *Difficulty:* because of the huge uncertainty, this problem is very difficult even to formulate in precise terms.
- *First aspect* of the problem: how to best distribute the sensors over the large area.
- *Status:* reasonable solutions exist for this aspect.
- *Second aspect* of the problem: what is the best location of each sensor in the corresponding zone.
- *This talk:* will focus on this aspect of the sensor placement problem.

72. Outline

- *Case study:* meteorological tower.
- *This case* is an example of multi-criteria optimization, when we need to maximize several objectives $x_1, \ldots, x_n$.
- *Traditional approach* to multi-objective optimization: maximize a weighted combination $\sum\limits_{i=1}^{n} w_i \cdot x_i$.
- *Specifics of our case:* constraints $x_i > x_i^{(0)}$ or $x_i < x_i^{(0)}$.
- *Equiv.:* $y_i > 0$, where $y_i \stackrel{\text{def}}{=} x_i - x_i^{(0)}$ or $y_i = x_i^{(0)} - x_i$.
- *Limitations* of using the traditional approach under constraints.
- *Scale invariance:* a brief description.
- *Main result:* scale invariance leads to a new approach: maximize $\sum\limits_{i=1}^{n} w_i \cdot \ln(y_i) = \sum\limits_{i=1}^{n} w_i \cdot \ln \left| x_i - x_i^{(0)} \right|$.

73. Case Study

- *Objective:* select the best location of a sophisticated multi-sensor meteorological tower.
- *Constraints:* we have several criteria to satisfy.
- *Example:* the station should not be located too close to a road.
- *Motivation:* the gas flux generated by the cars do not influence our measurements of atmospheric fluxes.
- *Formalization:* the distance x_1 to the road should be larger than a threshold t_1: $x_1 > t_1$, or $y_1 \stackrel{\text{def}}{=} x_1 - t_1 > 0$.
- *Example:* the inclination x_2 at the tower's location should be smaller than a threshold t_2: $x_2 < t_2$.
- *Motivation:* otherwise, the flux determined by this inclination and not by atmospheric processes.

74. General Case

- *In general:* we have several differences $y_1, \ldots, y_n$ all of which have to be non-negative.
- For each of the differences y_i, the larger its value, the better.
- Our problem is a typical setting for *multi-criteria optimization.*
- A most widely used approach to multi-criteria optimization is *weighted average*, where
  - we assign weights $w_1, \ldots, w_n > 0$ to different criteria y_i and
  - select an alternative for which the weighted average

$$w_1 \cdot y_1 + \ldots + w_n \cdot y_n$$

attains the largest possible value.

75. Limitations of the Weighted Average Approach

- *In general:* the weighted average approach often leads to reasonable solutions of the multi-criteria problem.
- *In our problem:* we have an additional requirement – that all the values y_i must be positive. So:
  - when selecting an alternative with the largest possible value of the weighted average,
  - we must only compare solutions with $y_i > 0$.
- *We will show:* under the requirement $y_i > 0$, the weighted average approach is not fully satisfactory.
- *Conclusion:* we need to find a more adequate solution.

76. Limitations of the Weighted Average Approach: Details

- The values y_i come from measurements, and measurements are never absolutely accurate.
- The results $\widetilde{y}_i$ of the measurements are not exactly equal to the actual (unknown) values y_i.
- *If:* for some alternative $y = (y_1, \ldots, y_n)$
  - we measure the values y_i with higher and higher accuracy and,
  - based on the measurement results $\widetilde{y}_i$, we conclude that y is better than some other alternative y'.
- *Then:* we expect that the actual alternative y is indeed better than y' (or at least of the same quality).
- Otherwise, we will not be able to make any meaningful conclusions based on real-life measurements.

77. The Above Natural Requirement Is Not Always Satisfied for Weighted Average

- *Simplest case:* two criteria y_1 and y_2, w/weights $w_i > 0$.
- If $y_1, y_2, y'_1, y'_2 > 0$, and $w_1 \cdot y_1 + w_2 \cdot y_2 > w_1 \cdot y'_1 + w_2 \cdot y'_2$, then $y = (y_1, y_2) \succ y' = (y'_1, y'_2)$.
- If $y_1 > 0$, $y_2 > 0$, and at least one of the values y'_1 and y'_2 is non-positive, then $y = (y_1, y_2) \succ y' = (y'_1, y'_2)$.
- Let us consider, for every $\varepsilon > 0$, the tuple $y(\varepsilon) \stackrel{\text{def}}{=} (\varepsilon, 1 + w_1/w_2)$, and $y' = (1, 1)$.
- In this case, for every $\varepsilon > 0$, we have

$$w_1 \cdot y_1(\varepsilon) + w_2 \cdot y_2(\varepsilon) = w_1 \cdot \varepsilon + w_2 + w_2 \cdot \frac{w_1}{w_2} = w_1 \cdot (1 + \varepsilon) + w_2$$

and $w_1 \cdot y'_1 + w_2 \cdot y'_2 = w_1 + w_2$, hence $y(\varepsilon) \succ y'$.

- However, in the limit $\varepsilon \to 0$, we have $y(0) = \left(0, 1 + \dfrac{w_1}{w_2}\right)$, with $y(0)_1 = 0$ and thus, $y(0) \prec y'$.

78. Towards a Precise Description

- Each alternative is characterized by a tuple of n positive values $y = (y_1, \ldots, y_n)$.
- Thus, the set of all alternatives is the set $(R^+)^n$ of all the tuples of positive numbers.
- For each two alternatives y and y', we want to tell whether
  - y is better than y' (we will denote it by $y \succ y'$ or $y' \prec y$),
  - or y' is better than y ($y' \succ y$),
  - or y and y' are equally good ($y' \sim y$).
- *Natural requirement:* if y is better than y' and y' is better than y'', then y is better than y''.
- The relation $\succ$ must be transitive.

79. Towards a Precise Description (cont-d)

- *Reminder:* the relation $\succ$ must be transitive.
- Similarly, the relation $\sim$ must be transitive, symmetric, and reflexive ($y \sim y$), i.e., be an *equivalence relation.*
- *An alternative description:* a transitive *pre-ordering* relation $a \succeq b \Leftrightarrow (a \succ b \vee a \sim b)$ s.t. $a \succeq b \vee b \succeq a$.
- Then, $a \sim b \Leftrightarrow (a \succeq b) \,\&\, (b \succeq a)$, and

$$a \succ b \Leftrightarrow (a \succeq b) \,\&\, (b \not\succeq a).$$

- *Additional requirement:*
  - *if* each criterion is better,
  - *then* the alternative is better as well.
- *Formalization:* if $y_i > y'_i$ for all i, then $y \succ y'$.

80. Scale Invariance: Motivation

- *Fact:* quantities y_i describe completely different physical notions, measured in completely different units.
- *Examples:* wind velocities measured in m/s, km/h, mi/h; elevations in m, km, ft.
- Each of these quantities can be described in many different units.
- A priori, we do not know which units match each other.
- Units used for measuring different quantities may not be exactly matched.
- It is reasonable to require that:
  - if we simply change the units in which we measure each of the corresponding n quantities,
  - the relations $\succ$ and $\sim$ between the alternatives $y = (y_1, \ldots, y_n)$ and $y' = (y'_1, \ldots, y'_n)$ do not change.

81. Scale Invariance: Towards a Precise Description

- *Situation:* we replace:
  - a unit in which we measure a certain quantity q
  - by a new measuring unit which is $\lambda > 0$ times smaller.
- *Result:* the numerical values of this quantity increase by a factor of λ: $q \to \lambda \cdot q$.
- *Example:* 1 cm is $\lambda = 100$ times smaller than 1 m, so the length $q = 2$ becomes $\lambda \cdot q = 2 \cdot 100 = 200$ cm.
- Then, scale-invariance means that for all $y, y' \in (R^+)^n$ and for all $\lambda_i > 0$, we have
  - $y = (y_1, \ldots, y_n) \succ y' = (y'_1, \ldots, y'_n)$ implies $(\lambda_1 \cdot y_1, \ldots, \lambda_n \cdot y_n) \succ (\lambda_1 \cdot y'_1, \ldots, \lambda_n \cdot y'_n)$,
  - $y = (y_1, \ldots, y_n) \sim y' = (y'_1, \ldots, y'_n)$ implies $(\lambda_1 \cdot y_1, \ldots, \lambda_n \cdot y_n) \sim (\lambda_1 \cdot y'_1, \ldots, \lambda_n \cdot y'_n)$.

82. Formal Description

- By a *total pre-ordering relation* on a set Y, we mean
  - a pair of a transitive relation $\succ$ and an equivalence relation $\sim$ for which,
  - for every $y, y' \in Y$, exactly one of the following relations hold: $y \succ y'$, $y' \succ y$, or $y \sim y'$.
- We say that a total pre-ordering is *non-trivial* if there exist y and y' for which $y \succ y'$.
- We say that a total pre-ordering relation on $(R^+)^n$ is:
  - *monotonic* if $y'_i > y_i$ for all i implies $y' \succ y$;
  - *continuous* if
    - whenever we have a sequence $y^{(k)}$ of tuples for which $y^{(k)} \succeq y'$ for some tuple y', and
    - the sequence $y^{(k)}$ tends to a limit y,
    - then $y \succeq y'$.

83. Main Result

Theorem. *Every non-trivial monotonic scale-inv. continuous total pre-ordering relation on* $(R^+)^n$ *has the form:*

$$y' = (y'_1, \ldots, y'_n) \succ y = (y_1, \ldots, y_n) \Leftrightarrow \prod_{i=1}^{n} (y'_i)^{\alpha_i} > \prod_{i=1}^{n} y_i^{\alpha_i};$$

$$y' = (y'_1, \ldots, y'_n) \sim y = (y_1, \ldots, y_n) \Leftrightarrow \prod_{i=1}^{n} (y'_i)^{\alpha_i} = \prod_{i=1}^{n} y_i^{\alpha_i},$$

for some constants $\alpha_i > 0$.

Comment: Vice versa,

- for each set of values $\alpha_1 > 0, \ldots, \alpha_n > 0$,
- the above formulas define a monotonic scale-invariant continuous pre-ordering relation on $(R^+)^n$.

84. Practical Conclusion

- *Situation:*
  - we need to select an alternative;
  - each alternative is characterized by characteristics $y_1, \ldots, y_n$.
- *Traditional approach:*
  - we assign the weights w_i to different characteristics;
  - we select the alternative with the largest value of $\sum\limits_{i=1}^n w_i \cdot y_i$.
- *New result:* it is better to select an alternative with the largest value of $\prod\limits_{i=1}^n y_i^{w_i}$.
- *Equivalent reformulation:* select an alternative with the largest value of $\sum\limits_{i=1}^n w_i \cdot \ln(y_i)$.

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Part VI
Proofs

86. Fair Price Under Twin Uncertainty: Proof

- In general, we have

$$([\underline{u},\overline{u}],[\underline{m},\overline{m}]) = ([\underline{m},\underline{m}],[\underline{m},\underline{m}])+([0,\overline{m}-\underline{m}],[0,\overline{m}-\underline{m}])+$$
$$([0,\overline{u}-\overline{m}],[0,0]) + ([\underline{u}-\underline{m},0],[0,0)].$$

- So, due to additivity:

$$P([\underline{u},\overline{u}],[\underline{m},\overline{m}]) = P([\underline{m},\underline{m}],[\underline{m},\underline{m}])+P([0,\overline{m}-\underline{m}],[0,\overline{m}-\underline{m}])+$$
$$P([0,\overline{u}-\overline{m}],[0,0]) + P([\underline{u}-\underline{m},0],[0,0)].$$

- Due to conservativeness, $P([\underline{m},\underline{m}],[\underline{m},\underline{m}]) = \underline{m}$.
- Similarly to the interval case, we can prove that:
  - $P([0,r],[0,r]) = \alpha_u \cdot r$ for some $\alpha_u \in [0,1]$,
  - $P([0,r],[0,0]) = \alpha_U \cdot r$ for some $\alpha_U \in [0,1]$;
  - $P([r,0],[0,0]) = \alpha_L \cdot r$ for some $\alpha_L \in [0,1]$.
- Thus,

$$P([\underline{u},\overline{u}],[\underline{m},\overline{m}]) = \underline{m}+\alpha_u\cdot(\overline{m}-\underline{m})+\alpha_U\cdot(\overline{U}-\overline{m})+\alpha_L\cdot(\underline{u}-\underline{m}).$$

87. Fuzzy Case: Proof

- Define $\mu_{\gamma,u}(0) = 1$, $\mu_{\gamma,u}(x) = \gamma$ for $x \in (0, u]$, and $\mu_{\gamma,u}(x) = 0$ for all other x.
- $\mathbf{s}_{\gamma,u}(\alpha) = [0, 0]$ for $\alpha > \gamma$, $\mathbf{s}_{\gamma,u}(\alpha) = [0, u]$ for $\alpha \le \gamma$.
- Based on the α-cuts, one check that $s_{\gamma,u+v} = s_{\gamma,u} + s_{\gamma,v}$.
- Thus, due to additivity, $P(s_{\gamma,u+v}) = P(s_{\gamma,u}) + P(s_{\gamma,v})$.
- Due to monotonicity, $P(s_{\gamma,u}) \uparrow$ when $u \uparrow$.
- Thus, $P(s_{\gamma,u}) = k^+(\gamma) \cdot u$ for some value $k^+(\gamma)$.
- Let us now consider a fuzzy number s s.t. $\mu(x) = 0$ for $x < 0$, $\mu(0) = 1$, then $\mu(x)$ continuously $\downarrow 0$.
- For each sequence of values $\alpha_0 = 1 < \alpha_1 < \alpha_2 < \ldots < \alpha_{n-1} < \alpha_n = 1$, we can form an approximation s_n:
  - $s_n^-(\alpha) = 0$ for all α; and
  - when $\alpha \in [\alpha_i, \alpha_{i+1})$, then $s_n^+(\alpha) = s^+(\alpha_i)$.

88. Fuzzy Case: Proof (cont-d)

- Here, $s_n = s_{\alpha_{n-1}, s^+(\alpha_{n-1})} + s_{\alpha_{n-2}, s^+(\alpha_{n-2}) - s^+(\alpha_{n-1})} + \ldots + s_{\alpha_1, \alpha_1 - \alpha_2}$.
- Due to additivity, $P(s_n) = k^+(\alpha_{n-1}) \cdot s^+(\alpha_{n-1}) + k^+(\alpha_{n-2}) \cdot (s^+(\alpha_{n-2}) - s^+(\alpha_{n-1})) + \ldots + k^+(\alpha_1) \cdot (\alpha_1 - \alpha_2)$.
- This is minus the integral sum for $\int_0^1 k^+(\gamma)\, ds^+(\gamma)$.
- Here, $s_n \to s$, so $P(s) = \lim P(s_n) = \int_0^1 k^+(\gamma)\, ds^+(\gamma)$.
- Similarly, for fuzzy numbers s with $\mu(x) = 0$ for $x > 0$, we have $P(s) = \int_0^1 k^-(\gamma)\, ds^-(\gamma)$ for some $k^-(\gamma)$.
- A general fuzzy number g, with α-cuts $[g^-(\alpha), g^+(\alpha)]$ and a point g_0 at which $\mu(g_0) = 1$, is the sum of g_0,
  - a fuzzy number with α-cuts $[0, g^+(\alpha) - g_0]$, and
  - a fuzzy number with α-cuts $[g_0 - g^-(\alpha), 0]$.
- Additivity completes the proof.

89. Case Study Proof: Part 1

- Due to scale-invariance, for every $y_1, \ldots, y_n$, $y'_1, \ldots, y'_n$, we can take $\lambda_i = \dfrac{1}{y_i}$ and conclude that

$$(y'_1, \ldots, y'_n) \sim (y_1, \ldots, y_n) \Leftrightarrow \left(\frac{y'_1}{y_1}, \ldots, \frac{y'_n}{y_n}\right) \sim (1, \ldots, 1).$$

- Thus, to describe the equivalence relation $\sim$, it is sufficient to describe $\{z = (z_1, \ldots, z_n) : z \sim (1, \ldots, 1)\}$.
- Similarly,

$$(y'_1, \ldots, y'_n) \succ (y_1, \ldots, y_n) \Leftrightarrow \left(\frac{y'_1}{y_1}, \ldots, \frac{y'_n}{y_n}\right) \succ (1, \ldots, 1).$$

- Thus, to describe the ordering relation $\succ$, it is sufficient to describe the set $\{z = (z_1, \ldots, z_n) : z \succ (1, \ldots, 1)\}$.
- Similarly, it is also sufficient to describe the set

$$\{z = (z_1, \ldots, z_n) : (1, \ldots, 1) \succ z\}.$$

90. Proof: Part 2

- *To simplify:* take logarithms $Y_i = \ln(y_i)$, and sets

$$S_\sim = \{Z : z = (\exp(Z_1), \ldots, \exp(Z_n)) \sim (1, \ldots, 1)\},$$
$$S_\succ = \{Z : z = (\exp(Z_1), \ldots, \exp(Z_n)) \succ (1, \ldots, 1)\};$$
$$S_\prec = \{Z : (1, \ldots, 1) \succ z = (\exp(Z_1), \ldots, \exp(Z_n))\}.$$

- Since the pre-ordering relation is total, for Z, either $Z \in S_\sim$ or $Z \in S_\succ$ or $Z \in S_\prec$.

- *Lemma:* $S_\sim$ is closed under addition:
  - $Z \in S_\sim$ means $(\exp(Z_1), \ldots, \exp(Z_n)) \sim (1, \ldots, 1)$;
  - due to scale-invariance, we have

$$(\exp(Z_1+Z_1'), \ldots) = (\exp(Z_1)\cdot\exp(Z_1'), \ldots) \sim (\exp(Z_1'), \ldots);$$

  - also, $Z' \in S_\sim$ means $(\exp(Z_1'), \ldots) \sim (1, \ldots, 1)$;
  - since $\sim$ is transitive,

$$(\exp(Z_1 + Z_1'), \ldots) \sim (1, \ldots) \text{ so } Z + Z' \in S_\sim.$$

91. Proof: Part 3

- *Reminder:* the set $S_\sim$ is closed under addition;
- Similarly, $S_\prec$ and $S_\succ$ are closed under addition.
- *Conclusion:* for every integer $q > 0$:
  - if $Z \in S_\sim$, then $q \cdot Z \in S_\sim$;
  - if $Z \in S_\succ$, then $q \cdot Z \in S_\succ$;
  - if $Z \in S_\prec$, then $q \cdot Z \in S_\prec$.
- Thus, if $Z \in S_\sim$ and $q \in N$, then $(1/q) \cdot Z \in S_\sim$.
- We can also prove that $S_\sim$ is closed under $Z \to -Z$:
  - $Z = (Z_1, \ldots) \in S_\sim$ means $(\exp(Z_1), \ldots) \sim (1, \ldots)$;
  - by scale invariance, $(1, \ldots) \sim (\exp(-Z_1), \ldots)$, i.e., $-Z \in S_\sim$.
- Similarly, $Z \in S_\succ \Leftrightarrow -Z \in S_\prec$.
- So $Z \in S_\sim \Rightarrow (p/q) \cdot Z \in S_\sim$; in the limit, $x \cdot Z \in S_\sim$.

92. Proof: Final Part

- *Reminder:* $S_\sim$ is closed under addition and multiplication by a scalar, so it is a linear space.
- *Fact:* $S_\sim$ cannot have full dimension n, since then all alternatives will be equivalent to each other.
- *Fact:* $S_\sim$ cannot have dimension $< n-1$, since then:
  - we can select an arbitrary $Z \in S_\prec$;
  - connect it w/$-Z \in S_\succ$ by a path γ that avoids $S_\sim$;
  - due to closeness, $\exists \gamma(t^*)$ in the limit of $S_\succ$ and $S_\prec$;
  - thus, $\gamma(t^*) \in S_\sim$ – a contradiction.
- Every $(n-1)$-dim lin. space has the form $\sum\limits_{i=1}^{n} \alpha_i \cdot Y_i = 0$.
- Thus, $Y \in S_\succ \Leftrightarrow \sum \alpha_i \cdot Y_i > 0$, and

$$y \succ y' \Leftrightarrow \sum \alpha_i \cdot \ln(y_i/y'_i) > 0 \Leftrightarrow \prod y_i^{\alpha_i} > \prod {y'_i}^{\alpha_i}.$$