Towards Designing Optimal Individualized Placement Tests

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1. Need for a Placement Test

- Computers enable us to provide individualized learning, at a pace tailored to each student.
- In order to start the learning process, it is important to find out the current level of the student's knowledge.
- Usually, such placement tests use a sequence of N problems of increasing complexity.
- If a student is able to solve a problem, the system generates a more complex one.
- If a student cannot solve a problem, the system generates an easier one, etc.
- Once we find the exact level of student's knowledge, the actual learning starts.
- It is desirable to get to actual leaning as soon as possible, i.e., to minimize the # of placement problems.

2. Bisection – Optimal Search Procedure

- At each stage, we have:
  - the largest level i at which a student can solve, &
  - the smallest level j at which s/he cannot.
- Initially, $i = 0$ (trivial), $j = N + 1$ (very tough).
- If $j = i + 1$, we found the student's level of knowledge.
- If $j > i + 1$, give a problem on level $m \stackrel{\text{def}}{=} (i + j)/2$:
  - if the student solved it, increase i to m;
  - else decrease j to m.
- In both cases, the interval $[i, j]$ is decreased by half.
- In s steps, we decrease the interval $[0, N + 1]$ to width $(N + 1) \cdot 2^{-s}$.
- In $s = \lceil \log_2(N + 1) \rceil$ steps, we get the interval of width ≤ 1, so the problem is solved.

3. Need to Account for Discouragement

- Every time a student is unable to solve a problem, he/she gets discouraged.
- In bisection, a student whose level is 0 will get $\approx \log_2(N+1)$ negative feedbacks.
- For positive answers, the student simply gets tired.
- For negative answers, the student also gets stressed and frustrated.
- If we count an effect of a positive answer as one, then the effect of a negative answer is $w > 1$.
- The value w can be individually determined.
- We need a testing scheme that minimizes the worst-case overall effect.

4. Analysis of the Problem

- We have $x = N + 1$ possible levels of knowledge.
- Let $e(x)$ denote the smallest possible effect needed to find out the student's knowledge level.
- We ask a student to solve a problem of some level n.
- If s/he solved it (effect $= 1$), we have $x - n$ possible levels $n, \ldots, N$.
- The effect of finding this level is $e(x - n)$, so overall effect is $1 + e(x - n)$.
- If s/he didn't (effect w), his/her level is between 0 and n, so we need effect $e(n)$, with overall effect $w + e(n)$.
- Overall worst-case effect is $\max(1 + e(x - n), w + e(n))$.
- In the optimal test, we select n for which this effect is the smallest, so $e(x) = \min\limits_{1 \le n < x} \max(1 + e(x - n), w + e(n))$.

5. Resulting Algorithm

- For $x = 1$, i.e., for $N = 0$, we have $e(1) = 0$.
- We know that $e(x) = \min\limits_{1 \le n < x} \max(1 + e(x - n), w + e(n))$.
- We can use this formula to sequentially compute the values $e(2)$, $e(3)$, $\ldots$, $e(N + 1)$.
- We also compute the corresponding minimizing values $n(2)$, $n(3)$, $\ldots$, $n(N + 1)$.
- Initially, $i = 0$ and $j = N + 1$.
- At each iteration, we ask to solve a problem at level $m = i + n(j - i)$:
  - if the student succeeds, we replace i with m;
  - else we replace j with m.
- We stop when $j = i + 1$; this means that the student's level is i.

6. Example 1: $N = 3$, $w = 3$

- Here, $e(1) = 0$.
- When $x = 2$, the only possible value for n is $n = 1$, so

$$e(2) = \min_{1 \le n < 2} \{\max\{1 + e(2-n), 3 + e(n)\}\} =$$

$$\max\{1 + e(1), 3 + e(1)\} = \max\{1, 3\} = 3.$$

- Here, $e(2) = 3$, and $n(2) = 1$.
- To find $e(3)$, we must compare two different values $n = 1$ and $n = 2$:

$$e(3) = \min_{1 \le n < 3} \{\max\{1 + e(3-n)), 3 + e(n)\}\} =$$

$$\min\{\max\{1+e(2), 3+e(1)\}, \max\{1+e(1), 3+e(2)\}\} =$$

$$\min\{\max\{4, 3\}, \max\{1, 6\}\} = \min\{4, 6\} = 4.$$

- Here, min is attained when $n = 1$, so $n(3) = 1$.

7. Example 1: $N = 3$, $w = 3$ (cont-d)

- To find $e(4)$, we must consider three possible values $n = 1$, $n = 2$, and $n = 3$, so

$$e(4) = \min_{1 \le n < 4} \{\max\{1 + e(4 - n), 3 + e(n))\}\} =$$

$$\min\{\max\{1 + e(3), 3 + e(1)\}, \max\{1 + e(2), 3 + e(2)\},$$

$$\max\{1 + e(1), 3 + e(3)\}\} =$$

$$\min\{\max\{5, 3\}, \max\{4, 6\}, \max\{1, 7\}\} =$$

$$\min\{5, 6, 7\} = 5.$$

- Here, min is attained when $n = 1$, so $n(4) = 1$.

8. Example 1: Resulting Procedure

- First, $i = 0$ and $j = 4$, so we ask a student to solve a problem at level $i + n(j - i) = 0 + n(4) = 1$.
- If the student fails level 1, his/her level is 0.
- If s/he succeeds at level 1, we set $i = 1$, and we assign a problem of level $1 + n(3) = 2$.
- If the student fails level 2, his/her level is 1.
- If s/he succeeds at level 2, we set $i = 2$, and we assign a problem of level $2 + n(3) = 3$.
- If the student fails level 3, his/her level is 2.
- If s/he succeeds at level 3, his/her level is 3.
- We can see that this is the most cautious scheme, when each student has at most one negative experience.

9. Example 2: $N = 3$ and $w = 1.5$

- We take $e(1) = 0$.
- When $x = 2$, then

$$e(2) = \min_{1 \le n < 2} \{\max\{1 + e(2-n), 3 + e(n)\}\} =$$

$$\max\{1 + e(1), 1.5 + e(1)\} = \max\{1, 1.5\} = 1.5.$$

- Here, $e(2) = 1.5$, and $n(2) = 1$.
- To find $e(3)$, we must compare two different values $n = 1$ and $n = 2$:

$$e(3) = \min_{1 \le n < 3} \{\max\{1 + e(3-n)), 1.5 + e(n)\}\} =$$

$$\min\{\max\{1+e(2), 1.5+e(1)\}, \max\{1+e(1), 1.5+e(2)\}\} =$$

$$\min\{\max\{2.5, 1.5\}, \max\{1, 3\}\} = \min\{2.5, 3\} = 2.5.$$

- Here, min is attained when $n = 1$, so $n(3) = 1$.

10. Example 2: $N = 3$ and $w = 1.5$ (cont-d)

- To find $e(4)$, we must consider three possible values $n = 1$, $n = 2$, and $n = 3$, so

$$e(4) = \min_{1 \le n < 4} \{\max\{1 + e(4 - n), 1.5 + e(n))\}\} =$$

$$\min\{\max\{1+e(3), 1.5+e(1)\}, \max\{1+e(2), 1.5+e(2)\},$$
$$\max\{1 + e(1), 1.5 + e(3)\}\} =$$
$$\min\{\max\{3.5, 1.5\}, \max\{2.5, 3\}, \max\{1, 4\}\} =$$
$$\min\{3.5, 3, 4\} = 3.$$

- Here, min is attained when $n = 2$, so $n(4) = 2$.

11. Example 2: Resulting Procedure

- First, $i = 0$ and $j = 4$, so we ask a student to solve a problem at level $i + n(j - i) = 0 + n(4) = 2$.
- If the student fails level 2, we set $j = 2$, and we assign a problem of level $0 + n(2) = 1$:
  - if the student fails level 1, his/her level is 0;
  - if s/he succeeds at level 1, his/her level is 1.
- If s/he succeeds at level 2, we set $i = 2$, and we assign a problem at level $2 + n(2) = 3$:
  - if the student fails level 3, his/her level is 2;
  - if s/he succeeds at level 3, his/her level is 3.
- We can see that in this case, the optimal testing scheme is bisection.

12. A Faster Algorithm May Be Needed

- For each n from 1 to N, we need to compare n different values.
- So, the total number of computational steps is proportional to $1 + 2 + \ldots + N = O(N^2)$.
- When N is large, N^2 may be too large.
- In some applications, the computation of the optimal testing scheme may takes too long.
- For this case, we have developed a faster algorithm for producing a testing scheme.
- The disadvantage of this algorithm is that it is only asymptotically optimal.

13. A Faster Algorithm for Generating an Asymptotically Optimal Testing Scheme

- First, we find the real number $\alpha \in [0,1]$ for which $\alpha + \alpha^w = 1$.
- This value α can be obtained, e.g., by applying bisection to the equation $\alpha + \alpha^w = 1$.
- At each iteration, once we know bounds i and j, we ask the student to solve a problem at the level
$$m = \lfloor \alpha \cdot i + (1-\alpha) \cdot j \rfloor.$$
- This algorithm is similar to bisection, expect that bisection corresponds to $\alpha = 0.5$.
- This makes sense, since for $w = 1$, the equation for α takes the form $2\alpha = 1$, hence $\alpha = 0.5$.
- For $w = 2$, the solution to the equation $\alpha + \alpha^2 = 1$ is the well-known golden ratio $\alpha = \dfrac{\sqrt{5}-1}{2} \approx 0.618$.

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