Is Alaska Negative-Tax Arrangement Fair? Almost: Mathematical Analysis

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1. What is negative tax and how it is arranged

- The US state of Alaska is one of the few places in the world where:
  - instead of paying taxes (i.e., paying money to the Government),
  - people receive a "negative tax" – an annual amount of money.
- At present, the negative tax arrangements are very straightforward.
- Every resident gets the exact same amount of money, irrespective of their other income.
- A poor person gets the same amount as a millionaire.

2. But is it fair?

- A natural question is: is this arrangement fair?
- On the one hand:
  - a millionaire does not need extra money, while
  - for a poor person, every dollar counts.
- So why not give the whole amount only to the poor folks?
- On the other hand, if we want to be fair, we may want to make sure that each person gets the same pleasure out of his/her money.
- To a poor person, receiving $1500 – this is an estimated 2024 per-person amount – is significant.
- For a millionaire it is barely noticeable.
- So should not we give more to richer people to make it more fair?
- After all, the usual taxes are proportional to the income.
- So why should not the negative tax be proportional to the income?

3. How this issue is usually discussed and what we do

- The issue of fairness of Alaska negative tax is usually discussed on the qualitative ethical level.
- This is typical for finance-related issues.
- In this talk, we provide a mathematical analysis of the problem.
- As a result of this analysis, we show that the current Alaska negative tax arrangement is almost fair.
- To be more precise, we show – honestly, somewhat contrary to our own intuition – that:
  - in the fair arrangement, the amount should slightly increase with income,
  - but increase very slowly – so that the richest person gets twice the amount of the poorest one.

4. How this issue is usually discussed and what we do (cont-d)

- From this viewpoint, the current arrangement when everyone gets the same amount is closer to the optimal distribution than the proportional idea:
  - in the actual arrangement, the richest person gets the same amount as the poorest person,
  - in the optimal arrangement, the richest person gets twice as much as the poorest person, while
  - in the proportional arrangements, the richest person would get thousands of times more than the poorest person.

5. What we mean by fair

- The problem of distributing the excess income is a particular case of the general problem of cooperative decision making, when:
  - we start with the status quo state, and
  - we compare different alternatives (each of which is better, for all participants, than the status quo).
- Such situations have been analyzed in the 1950s by the (future Nobelist) John Nash in the framework of decision theory.
- According to decision theory, preferences of a rational person can be described by a special function – called *utility function*.
- This function assigns, to each alternative A, a number $u(A)$ such that:
  - the person prefers A to B if and only if
  - the utility $u(A)$ is larger than the utility $u(B)$.

6. What we mean by fair (cont-d)

- Utility is usually defined in such a way that:
  - if we have an alternative A that leads to outcomes A_i with probabilities p_i,
  - then the utility of A is equal to $u(A) = p_1 \cdot u(A_1) + \ldots + p_n \cdot u(A_n)$.
- It is known that these conditions define the utility function modulo an increasing linear transformation.
- Namely:
  - if the function $u(A)$ correctly describes the person's preferences,
  - then, for each values c_0 and $c_1 > 0$, the function $v(A) \stackrel{\text{def}}{=} c_0 + c_1 \cdot u(A)$ describes the same preferences.
- Also:
  - if two functions $u(A)$ and $v(A)$ describe the same preferences,
  - then there exist real numbers c_0 and $c_1 > 0$ for which $v(A) = c_0 + c_1 \cdot u(A)$ for all A.

7. What we mean by fair (cont-d)

- In the cooperative decision making, we have N agents with utility functions $u_1(A)$, ..., $u_N(A)$.
- We have a fixed status quo state A_0.
- So, we can replace each original utility function with an equivalent function $U_i(A) \stackrel{\text{def}}{=} u_i(A) - u_i(A_0)$ for which $U_i(A_0) = 0$.
- With this restriction, the utility functions are still not uniquely determined.
- For each i and for each value c_i, we can still replace:
  - the original utility function $U_i(A)$ with
  - an equivalent re-scaled function $c_i \cdot U_i(A)$ that describes the same preferences.
- Based on the values $U_1(A), \ldots, U_N(A)$ corresponding to different alternatives A, we must decide which alternative is better.

8. What we mean by fair (cont-d)

- It makes sense to require that our choice should not depend on renaming the participants.
- It also makes sense to require that the selection should not change if we replace each utility function $U_i(A)$ with an equivalent one $c_i \cdot U_i(A)$.
- It also makes sense to require that if for all participants A is better than B, then out of two options A and B the group should select A.
- Nash has proven than:
  - under these reasonable conditions,
  - the group should select the alternative for which the product of the utilities $U_1(A) \cdot \ldots \cdot U_N(A)$ is the largest possible.
- This is known as *Nash's bargaining solution.*
- This is what we will use to describe a fair solution.

9. Let us apply Nash's bargaining solution to our problem

- To apply Nash's bargaining solution to our problem, we need to recall how utility depends on money.
- This is *not* a linear dependence.
- As we have mentioned earlier, an extra \$1500 means a lot to a poor person and practically nothing to a millionaire.
- Empirical analysis shows that the utility is proportional to the square root of the amount of money x: $u(x) = k \cdot \sqrt{x}$, for some coefficient $k > 0$.
- Let v_i denote the original income of the i-th person – before the negative tax.
- This means that at the status quo state, the i-th person has utility $u_i(A_0) = k_i \cdot \sqrt{v_i}$, for some k_i.
- If we give an additional amount t_i to the i-th person, then his/her utility becomes equal to $u_i(A) = k_i \cdot \sqrt{v_i + t_i}$.

10. Let us apply Nash's bargaining solution to our problem (cont-d)

- So, the re-scaled utility value – for which the utility of the status quo alternative is 0 – is equal to

$$U_i = u_i(A_i) - u_i(A_0) = k_i \cdot \sqrt{v_i + t_i} - k_i \cdot \sqrt{v_i} = k_i \cdot \left(\sqrt{v_i + t_i} - \sqrt{v_i}\right).$$

- So:
  - if we denote the overall amount of the money to be distributed by $T = t_1 + \ldots + t_N$,
  - then the Nash's bargaining solution takes the following form.

11. Mathematical formulation of the problem

- We are given the value $T > 0$ and the non-negative values $v_1 \geq 0$, $\ldots$, $v_N > 0$.
- We consider all the tuples $t_1 \geq 0$, $\ldots$, $t_n > 0$ that satisfy the constraint

$$t_1 + \ldots + t_N = T.$$

- Between them, we must find the tuple for which the following product if the largest possible:

$$k_1 \cdot (\sqrt{v_1 + t_1} - \sqrt{v_1}) \cdot \ldots \cdot k_N \cdot \left(\sqrt{v_N + t_N} - \sqrt{v_N}\right).$$

12. Let us make the problem somewhat simpler

- Let us first notice that if $a > b$ and we multiply both values by the same positive constant k, we still have $k \cdot a > k \cdot b$.
- Similarly, inequalities do not change if we divide both sides by the same positive number.
- Thus:
  - if we divide all the values of the objective function by a positive number $k_1 \cdot \ldots \cdot k_N$,
  - this will not change which tuples have a larger value of this function and which have smaller value.
- Thus, instead of maximizing the original product, we can maximize a simpler expression

$$\left(\sqrt{v_1 + t_1} - \sqrt{v_1}\right) \cdot \ldots \cdot \left(\sqrt{v_N + t_N} - \sqrt{v_N}\right).$$

- This objective function is a product.

13. Let us make the problem somewhat simpler (cont-d)

- From the computational viewpoint, a product is somewhat more complex than a sum.
- It is known how to reduce a product to a sum – this is what logarithms were invented for.
- The function $\ln(x)$ is strictly increasing, so maximizing the objective function is equivalent to maximizing its logarithm.
- Since the logarithm of the product is equal to the product of logarithms, we get the following equivalent problem.
- Under the constraint $t_1 + \ldots + t_N = T$, maximize the following expression:

$$\ln\left(\sqrt{v_1 + t_1} - \sqrt{v_1}\right) + \ldots + \ln\left(\sqrt{v_N + t_N} - \sqrt{v_N}\right).$$

14. Let us solve the problem

- To solve the constraint optimization problem, we can use the usual Lagrange multiplier method.
- Thus, we reduce it to the following unconstrained optimization problem: maximize the expression

$$\ln\left(\sqrt{v_1+t_1}-\sqrt{v_1}\right)+\ldots+\ln\left(\sqrt{v_N+t_N}-\sqrt{v_N}\right)+\lambda\cdot(t_1+\ldots+t_N-T).$$

- Here, the coefficient λ needs to be determined.
- To find the minimum of the resulting expression, we:
  - differentiate it with respect to each unknown t_i and
  - equate the resulting derivative to 0.
- As a result, we get the following equality:

$$\frac{1}{\sqrt{v_i+t_i^{\text{opt}}}-\sqrt{v_i}}\cdot\frac{1}{2\cdot\sqrt{v_i+t_i^{\text{opt}}}}+\lambda=0.$$

15. Let us solve the problem (cont-d)

- Let us:
  - multiply the two fractions by multiplying their numerators and denominators and
  - take into account that the product of two square roots is the original value.
- So, we conclude that

$$\frac{1}{2\cdot\left(v_i+t_i^{\rm opt}-\sqrt{v_i\cdot\left(v_i+t_i^{\rm opt}\right)}\right)}+\lambda=0.$$

- If we multiply both sides of this equality by 2 and move the resulting term 2λ to the right-hand side, we get

$$\frac{1}{v_i+t_i^{\rm opt}-\sqrt{v_i\cdot\left(v_i+t_i^{\rm opt}\right)}}=-2\lambda.$$

16. Let us solve the problem (cont-d)

- If we now take an inverse of both sides, we get

$$v_i + t_i^{\rm opt} - \sqrt{v_i \cdot \left(v_i + t_i^{\rm opt}\right)} = t_0.$$

- Here we denoted

$$t_0 \stackrel{\rm def}{=} -\frac{1}{2\lambda}.$$

17. What will happen in extreme cases?

- Before we consider the general case, let us analyze what will happen in the two extreme vases:
  - of a poorest person for whom $v_i = 0$ and
  - of the richest person for whom $v_i \to \infty$.
- For the poorest person case, when $v_i = 0$, the above equation leads to $t_i^{\text{opt}} = t_0$.
- For the richest person case when $v_i \to \infty$, we have

$$\sqrt{v_i \cdot \left(v_i + t_i^{\text{opt}}\right)} = \sqrt{v_i^2 \cdot \left(1 + \frac{t_i^{\text{opt}}}{v_i}\right)} = v_i \cdot \sqrt{1 + \frac{t_i^{\text{opt}}}{v_i}}.$$

- The value t_i is bounded by T while v_i tends to infinity.
- Thus, the ratio t_i^{opt}/v_i tends to 0.
- In general,

$$\sqrt{1+\varepsilon} = 1 + \frac{1}{2} \cdot \varepsilon + O(\varepsilon^2).$$

18. What will happen in extreme cases (cont-d)

- Thus, we get

$$\sqrt{v_i \cdot \left(v_i + t_i^{\text{opt}}\right)} = v_i \cdot \left(1 + \frac{t_i^{\text{opt}}}{2v_i} + O\left(\left(\frac{t_i^{\text{opt}}}{2v_i}\right)^2\right)\right) = v_i + \frac{t_i^{\text{opt}}}{2} + o(1).$$

- Thus, in the limit $v_i \to \infty$, the equation takes the form

$$v_i + t_i^{\text{opt}} - v_i - \frac{t_i^{\text{opt}}}{2} = t_0, \text{ i.e., } \frac{t_i^{\text{opt}}}{2} = t_0.$$

- So, $t_i^{\text{opt}} = 2t_0$: in the Nash's fair arrangement, the richest person indeed gets twice as much as the poorest person.

- We prove that in the general case, the solution t_i^{opt} is always between t_0 and $2t_0$.

19. How can we actually compute the fair solution?

- We have proved thar, once we know t_0, we can explicitly compute all the values $t_i^{\rm opt}$ by using a straightforward formula

$$t_i^{\rm opt}=t_0-\frac{v_i}{2}+\sqrt{\frac{v_i^2}{4}+v_i\cdot t_0}.$$

- The value t_0 can then be found if we substitute these expressions for into the formula $t_i^{\rm opt}$ into the formula

$$t_1^{\rm opt}+\ldots+t_N^{\rm opt}=T.$$

- Thus, get the following equation with one unknown (which is, thus, easy to solve):

$$\sum_{i=1}^N\left(t_0-\frac{v_i}{2}+\sqrt{\frac{v_i^2}{4}+v_i\cdot t_0}\right)=T.$$

20. Comment

- Let us show that our formula agrees with both extreme cases mentioned above.
- Indeed, for $v_i = 0$, we clearly have $t_i^{\text{opt}} = t_0$.
- For $v_i \to \infty$, we have

$$\sqrt{\frac{v_i^2}{4} + v_i \cdot t_0} = \sqrt{\frac{v_i^2}{4} \cdot \left(1 + \frac{4t_0}{v_i}\right)} = \frac{v_i}{2} \cdot \left(1 + \frac{2t_0}{v_i} + o\right) = \frac{v_i}{2} + t_0 + o(1).$$

- Thus, the above expression takes the following form:

$$t_i^{\text{opt}} = t_0 - \frac{v_i}{2} + \frac{v_i}{2} + t_0 = o(1) = 2t_0 + o(1).$$

- So, in the limit, we indeed get $t_i^{\text{opt}} = 2t_0$.

21. Proof that the optimal gain is always between t_0 and $2t_0$

- For $v_i = 0$, the above equation leads to $t_i^{\text{opt}} = t_0$.
- Thus, to prove the desired statement, it is sufficient to consider the case when $v_i > 0$.
- Let us first prove, by contradiction, that we cannot have $t_i^{\text{opt}} = 0$ for some i.
- Indeed, in this case, the corresponding utility U_i is 0, so the product of utilities is 0.
- Thus, it cannot be the largest possible value.
- Indeed, if we simply divide $T > 0$ into N equal parts, we get all utilities positive – and thus, the positive product of utilities.

22. Proof: Part 2

- Let us now prove that $t_0 > 0$.
- Due to our equation, the desired inequality is equivalent to

$$v_i + t_i^{\text{opt}} - \sqrt{v_i \cdot \left(v_i + t_i^{\text{opt}}\right)} > 0.$$

- This is, in turn, equivalent to $v_i + t_i^{\text{opt}} > \sqrt{v_i \cdot \left(v_i + t_i^{\text{opt}}\right)}$.
- Both sides are non-negative.
- For non-negative numbers, the function $x \mapsto x^2$ is strictly increasing.
- So, the last inequality is equivalent to what we will get by squaring both sides: $v_i^2 + 2v_i \cdot t_i^{\text{opt}} + \left(t_i^{\text{opt}}\right)^2 > v_i^2 + v_i \cdot t_i^{\text{opt}}$.
- Subtracting the right-hand side from the left-hand side, we get the equivalent inequality $v_i \cdot t_i^{\text{opt}} + \left(t_i^{\text{opt}}\right)^2 > 0$.
- This inequality is clearly true, since $v_i \geq 0$ and $t_i^{\text{opt}} > 0$.
- Thus, the original inequality $t_0 > 0$ is also true.

23. Proof: Part 3

- In our main equation, if we move t_0 to the left-hand side, we get an equivalent equation $L\left(t_i^{\text{opt}}\right) = 0$, where we denoted

$$L(t_i) \stackrel{\text{def}}{=} v_i + t_i - \sqrt{v_i \cdot (v_i + t_i)} - t_0.$$

- Let us prove that $L(t_i)$ is a strictly increasing function of t_i.
- For this purpose, it is sufficient to prove that the partial derivative of $L(t_i)$ with respect to t_i is always positive.
- Here, $\dfrac{\partial L}{\partial t_i} = 1 - \dfrac{v_i}{2\sqrt{v_i \cdot (v_i + t_i)}}$.
- Here, $v_i \cdot (v_i + t_i) \geq v_i^2$, thus $v_i \leq \sqrt{v_i \cdot (v_i + t_i)}$.
- Thus,

$$\frac{v_i}{2\sqrt{v_i \cdot (v_i + t_i)}} \leq \frac{1}{2} \text{ and } \frac{\partial L}{\partial t_i} \geq 1 - \frac{1}{2} = \frac{1}{2} > 0.$$

- The statement is proven.

24. Proof: Part 4

- Let us now prove that for $t_i = t_0$, we have $L(t_0) < 0$ (remember that we assumed that $v_i > 0$).
- Indeed, the desired inequality has the form

$$v_i + t_0 - \sqrt{v_i \cdot (v_i + t_0)} - t_0 = v_i - \sqrt{v_i \cdot (v_i + t_0)} < 0.$$

- This is equivalent to $v_i < \sqrt{v_i \cdot (v_i + t_0)}$.
- Here, both sides are non-negative, so we can get an equivalent inequality by squaring both sides:

$$v_i^2 < v_i \cdot (v_i + t_0) = v_i^2 + v_i \cdot t_0.$$

- By subtracting v_i^2 from both sides, we get $0 < v_i \cdot t_0$.
- This is clearly true since $v_i > 0$ and $t_0 > 0$.
- Thus, the equivalent inequality $L < 0$ is true too.

25. Proof: Part 5

- Let us now prove that for $t_i = 2t_0$, we have $L(2t_0) > 0$.
- Indeed, the desired inequality has the form

$$v_i + 2t_0 - \sqrt{v_i \cdot (v_i + 2t_0)} - t_0 = v_i + t_0 - \sqrt{v_i \cdot (v_i + 2t_0)} > 0.$$

- This is equivalent to $v_i + t_0 > \sqrt{v_i \cdot (v_i + 2t_0)}$.
- Here, both sides are non-negative.
- So we can get an equivalent inequality by squaring both sides:

$$v_i^2 + 2v_i \cdot t_0 + t_0^2 > v_i \cdot (v_i + t_0) = v_i^2 + v_i \cdot t_0.$$

- By subtracting $v_i^2 + 2v_i \cdot t_0$ from both sides, we get $t_0^2 > 0$.
- This is clearly true since $t_0 > 0$.
- Thus, the equivalent inequality $L > 0$ is true too.

26. Proof: Part 6

- Now, we are ready to prove the proposition.
- Indeed, according to Part 3 of this proof, the function $L(t_i)$ is strictly increasing.
- It is negative for $t_i = t_0$, it is positive for $t_i = 2t_0$.
- So the value t_i^{opt} for which $L\left(t_i^{\text{opt}}\right) = 0$ must indeed be between t_0 and $2t_0$.
- The desired statement is proven.

27. Proof of the formula

- Let us move the square root to the right-hand side of the formula and t_0 to the left-hand side.
- Then, we get the following formula:

$$v_i + t_i^{\rm opt} - t_0 = \sqrt{v_i \cdot \left(v_i + t_i^{\rm opt}\right)}.$$

- Squaring both sides and opening the parentheses in the right-hand side, we get

$$v_i^2 + 2v_i \cdot t_i^{\rm opt} - 2v_i \cdot t_0 + \left(t_i^{\rm opt}\right)^2 - 2t_i^{\rm opt} \cdot t_0 + t_0^2 = v_i^2 + v_i \cdot t_i^{\rm opt}.$$

- Let us subtract v_i^2 from both sides, move all the terms to the left-hand side, and combine terms proportional to $t_i^{\rm opt}$.
- Then, we get the following quadratic equation for determining $t_i^{\rm opt}$:

$$\left(t_i^{\rm opt}\right)^2 + t_i^{\rm opt} \cdot (v_i - 2t_0) + (t_0^2 - 2v_i \cdot t_0) = 0.$$

28. Proof of the formula (cont-d)

- For an equation $x^2 + p \cdot x + q = 0$, the general solution is

$$x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

- For our equation, this leads to

$$t_i^{\text{opt}} = t_0 - \frac{v_i}{2} \pm \sqrt{\left(t_0 - \frac{v_i}{2}\right)^2 - t_0^2 + 2v_i \cdot t_0}.$$

- The expression under the square root is equal to

$$t_0^2 - v_i \cdot t_0 + \frac{v_i^2}{4} - t_0^2 + 2v_i \cdot t_0.$$

- The terms proportional to t_0^2 cancel each other, and the terms $-v_i \cdot t_0$ and $2v_i \cdot t_0$ lead to $v_i \cdot t_0$.

- Thus, the expression under the square root is equal to

$$\frac{v_i^2}{4} + v_i \cdot t_0.$$

29. Proof of the formula (cont-d)

- So, the last formula takes the form

$$t_i^{\text{opt}} = t_0 - \frac{v_i}{2} \pm \sqrt{\frac{v_i^2}{4} + v_i \cdot t_0}.$$

- We have proven that t_i^{opt} is always greater than or equal to t_0.
- Thus, we cannot have the minus sign in this formula.
- So, we must have plus.
- Hence, we get the desired formula.

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