Is Alaska Negative-Tax Arrangement Fair? Almost: Mathematical Analysis

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1. What is negative tax and how it is arranged

- The US state of Alaska is one of the few places in the world where:
  - instead of paying taxes (i.e., paying money to the Government),
  - people receive a “negative tax” – an annual amount of money.
- At present, the negative tax arrangements are very straightforward.
- Every resident gets the exact same amount of money, irrespective of their other income.
- A poor person gets the same amount as a millionaire.
2. But is it fair?

- A natural question is: is this arrangement fair?
- On the one hand:
  - a millionaire does not need extra money, while
  - for a poor person, every dollar counts.
- So why not give the whole amount only to the poor folks?
- On the other hand, if we want to be fair, we may want to make sure that each person gets the same pleasure out of his/her money.
- To a poor person, receiving $1500 – this is an estimated 2024 per-person amount – is significant.
- For a millionaire it is barely noticeable.
- So should not we give more to richer people to make it more fair?
- After all, the usual taxes are proportional to the income.
- So why should not the negative tax be proportional to the income?
3. How this issue is usually discussed and what we do

- The issue of fairness of Alaska negative tax is usually discussed on the qualitative ethical level.
- This is typical for finance-related issues.
- In this talk, we provide a mathematical analysis of the problem.
- As a result of this analysis, we show that the current Alaska negative tax arrangement is almost fair.
- To be more precise, we show – honestly, somewhat contrary to our own intuition – that:
  - in the fair arrangement, the amount should slightly increase with income,
  - but increase very slowly – so that the richest person gets twice the amount of the poorest one.
4. How this issue is usually discussed and what we do (cont-d)

- From this viewpoint, the current arrangement when everyone gets the same amount is closer to the optimal distribution than the proportional idea:
  - in the actual arrangement, the richest person gets the same amount as the poorest person,
  - in the optimal arrangement, the richest person gets twice as much as the poorest person, while
  - in the proportional arrangements, the richest person would get thousands of times more than the poorest person.
5. What we mean by fair

- The problem of distributing the excess income is a particular case of the general problem of cooperative decision making, when:
  - we start with the status quo state, and
  - we compare different alternatives (each of which is better, for all participants, than the status quo).

- Such situations have been analyzed in the 1950s by the (future Nobelist) John Nash in the framework of decision theory.

- According to decision theory, preferences of a rational person can be described by a special function – called *utility function*.

- This function assigns, to each alternative $A$, a number $u(A)$ such that:
  - the person prefers $A$ to $B$ if and only if
  - the utility $u(A)$ is larger than the utility $u(B)$. 
6. What we mean by fair (cont-d)

- Utility is usually defined in such a way that:
  - if we have an alternative $A$ that leads to outcomes $A_i$ with probabilities $p_i$,
  - then the utility of $A$ is equal to $u(A) = p_1 \cdot u(A_1) + \ldots + p_n \cdot u(A_n)$.

- It is known that these conditions define the utility function modulo an increasing linear transformation.

- Namely:
  - if the function $u(A)$ correctly describes the person’s preferences,
  - then, for each values $c_0$ and $c_1 > 0$, the function $v(A) \overset{\text{def}}{=} c_0 + c_1 \cdot u(A)$ describes the same preferences.

- Also:
  - if two functions $u(A)$ and $v(A)$ describe the same preferences,
  - then there exist real numbers $c_0$ and $c_1 > 0$ for which $v(A) = c_0 + c_1 \cdot u(A)$ for all $A$. 
7. What we mean by fair (cont-d)

- In the cooperative decision making, we have $N$ agents with utility functions $u_1(A), \ldots, u_N(A)$.
- We have a fixed status quo state $A_0$.
- So, we can replace each original utility function with an equivalent function $U_i(A) \overset{\text{def}}{=} u_i(A) - u_i(A_0)$ for which $U_i(A_0) = 0$.
- With this restriction, the utility functions are still not uniquely determined.
- For each $i$ and for each value $c_i$, we can still replace:
  - the original utility function $U_i(A)$ with
  - an equivalent re-scaled function $c_i \cdot U_i(A)$ that describes the same preferences.
- Based on the values $U_1(A), \ldots, U_N(A)$ corresponding to different alternatives $A$, we must decide which alternative is better.
8. What we mean by fair (cont-d)

- It makes sense to require that our choice should not depend on re-naming the participants.
- It also makes sense to require that the selection should not change if we replace each utility function $U_i(A)$ with an equivalent one $c_i \cdot U_i(A)$.
- It also makes sense to require that if for all participants $A$ is better than $B$, then out of two options $A$ and $B$ the group should select $A$.
- Nash has proven than:
  - under these reasonable conditions,
  - the group should select the alternative for which the product of the utilities $U_1(A) \cdot \ldots \cdot U_N(A)$ is the largest possible.
- This is known as Nash’s bargaining solution.
- This is what we will use to describe a fair solution.
9. Let us apply Nash’s bargaining solution to our problem

- To apply Nash’s bargaining solution to our problem, we need to recall how utility depends on money.
- This is not a linear dependence.
- As we have mentioned earlier, an extra $1500 means a lot to a poor person and practically nothing to a millionaire.
- Empirical analysis shows that the utility is proportional to the square root of the amount of money $x$: $u(x) = k \cdot \sqrt{x}$, for some coefficient $k > 0$.
- Let $v_i$ denote the original income of the $i$-th person – before the negative tax.
- This means that at the status quo state, the $i$-th person has utility $u_i(A_0) = k_i \cdot \sqrt{v_i}$, for some $k_i$.
- If we give an additional amount $t_i$ to the $i$-th person, then his/her utility becomes equal to $u_i(A) = k_i \cdot \sqrt{v_i + t_i}$. 
10. Let us apply Nash’s bargaining solution to our problem (cont-d)

- So, the re-scaled utility value – for which the utility of the status quo alternative is 0 – is equal to

\[ U_i = u_i(A_i) - u_i(A_0) = k_i \cdot \sqrt{v_i + t_i} - k_i \cdot \sqrt{v_i} = k_i \cdot (\sqrt{v_i + t_i} - \sqrt{v_i}). \]

- So:
  - if we denote the overall amount of the money to be distributed by \( T = t_1 + \ldots + t_N \),
  - then the Nash’s bargaining solution takes the following form.
11. Mathematical formulation of the problem

- We are given the value $T > 0$ and the non-negative values $v_1 \geq 0$, $\ldots$, $v_N > 0$.

- We consider all the tuples $t_1 \geq 0$, $\ldots$, $t_n > 0$ that satisfy the constraint
  \[ t_1 + \ldots + t_N = T. \]

- Between them, we must find the tuple for which the following product is the largest possible:
  \[ k_1 \cdot (\sqrt{v_1 + t_1} - \sqrt{v_1}) \cdot \ldots \cdot k_N \cdot (\sqrt{v_N + t_N} - \sqrt{v_N}). \]
12. Let us make the problem somewhat simpler

- Let us first notice that if \( a > b \) and we multiply both values by the same positive constant \( k \), we still have \( k \cdot a > k \cdot b \).

- Similarly, inequalities do not change if we divide both sides by the same positive number.

- Thus:
  - if we divide all the values of the objective function by a positive number \( k_1 \cdot \ldots \cdot k_N \),
  - this will not change which tuples have a larger value of this function and which have smaller value.

- Thus, instead of maximizing the original product, we can maximize a simpler expression

\[
(\sqrt{v_1 + t_1} - \sqrt{v_1}) \cdot \ldots \cdot (\sqrt{v_N + t_N} - \sqrt{v_N})
\]

- This objective function is a product.
13. Let us make the problem somewhat simpler (cont-d)

- From the computational viewpoint, a product is somewhat more complex than a sum.

- It is known how to reduce a product to a sum – this is what logarithms were invented for.

- The function \( \ln(x) \) is strictly increasing, so maximizing the objective function is equivalent to maximizing its logarithm.

- Since the logarithm of the product is equal to the product of logarithms, we get the following equivalent problem.

- Under the constraint \( t_1 + \ldots + t_N = T \), maximize the following expression:

\[
\ln \left( \sqrt{v_1 + t_1} - \sqrt{v_1} \right) + \ldots + \ln \left( \sqrt{v_N + t_N} - \sqrt{v_N} \right).
\]
14. Let us solve the problem

- To solve the constraint optimization problem, we can use the usual Lagrange multiplier method.
- Thus, we reduce it to the following unconstrained optimization problem: maximize the expression
  \[
  \ln (\sqrt{v_1 + t_1} - \sqrt{v_1}) + \ldots + \ln (\sqrt{v_N + t_N} - \sqrt{v_N}) + \lambda \cdot (t_1 + \ldots + t_N - T).
  \]
- Here, the coefficient \( \lambda \) needs to be determined.
- To find the minimum of the resulting expression, we:
  - differentiate it with respect to each unknown \( t_i \) and
  - equate the resulting derivative to 0.
- As a result, we get the following equality:
  \[
  \frac{1}{\sqrt{v_i + t_i^{\text{opt}}} - \sqrt{v_i}} \cdot \frac{1}{2 \cdot \sqrt{v_i + t_i^{\text{opt}}}} + \lambda = 0.
  \]
15. Let us solve the problem (cont-d)

- Let us:
  - multiply the two fractions by multiplying their numerators and denominators and
  - take into account that the product of two square roots is the original value.

- So, we conclude that

\[
\frac{1}{2 \cdot \left( v_i + t_i^{opt} - \sqrt{v_i \cdot (v_i + t_i^{opt})} \right)} + \lambda = 0.
\]

- If we multiply both sides of this equality by 2 and move the resulting term \(2\lambda\) to the right-hand side, we get

\[
\frac{1}{v_i + t_i^{opt} - \sqrt{v_i \cdot (v_i + t_i^{opt})}} = -2\lambda.
\]
16. Let us solve the problem (cont-d)

- If we now take an inverse of both sides, we get

\[ v_i + t_i^{\text{opt}} - \sqrt{v_i \cdot (v_i + t_i^{\text{opt}})} = t_0. \]

- Here we denoted

\[ t_0 \overset{\text{def}}{=} - \frac{1}{2\lambda}. \]
17. What will happen in extreme cases?

- Before we consider the general case, let us analyze what will happen in the two extreme vases:
  - of a poorest person for whom $v_i = 0$ and
  - of the richest person for whom $v_i \to \infty$.

- For the poorest person case, when $v_i = 0$, the above equation leads to $t_i^{\text{opt}} = t_0$.

- For the richest person case when $v_i \to \infty$, we have
  \[ \sqrt{v_i \cdot (v_i + t_i^{\text{opt}})} = \sqrt{v_i^2 \cdot \left(1 + \frac{t_i^{\text{opt}}}{v_i}\right)} = v_i \cdot \sqrt{1 + \frac{t_i^{\text{opt}}}{v_i}}. \]

- The value $t_i$ is bounded by $T$ while $v_i$ tends to infinity.
- Thus, the ratio $t_i^{\text{opt}}/v_i$ tends to 0.
- In general,
  \[ \sqrt{1 + \varepsilon} = 1 + \frac{1}{2} \cdot \varepsilon + O(\varepsilon^2). \]
Thus, we get
\[ \sqrt{v_i \cdot (v_i + t_i^{opt})} = v_i \cdot \left( 1 + \frac{t_i^{opt}}{2v_i} + O \left( \left( \frac{t_i^{opt}}{2v_i} \right)^2 \right) \right) = v_i + \frac{t_i^{opt}}{2} + o(1). \]

Thus, in the limit \( v_i \to \infty \), the equation takes the form
\[ v_i + t_i^{opt} - v_i - \frac{t_i^{opt}}{2} = t_0, \quad \text{i.e.,} \quad \frac{t_i^{opt}}{2} = t_0. \]

So, \( t_i^{opt} = 2t_0 \): in the Nash’s fair arrangement, the richest person indeed gets twice as much as the poorest person.

We prove that in the general case, the solution \( t_i^{opt} \) is always between \( t_0 \) and \( 2t_0 \).
19. How can we actually compute the fair solution?

- We have proved that, once we know $t_0$, we can explicitly compute all the values $t_i^{\text{opt}}$ by using a straightforward formula

$$t_i^{\text{opt}} = t_0 - \frac{v_i}{2} + \sqrt{\frac{v_i^2}{4} + v_i \cdot t_0}.$$

- The value $t_0$ can then be found if we substitute these expressions for $t_i^{\text{opt}}$ into the formula $t_1^{\text{opt}} + \ldots + t_N^{\text{opt}} = T$.

- Thus, get the following equation with one unknown (which is, thus, easy to solve):

$$\sum_{i=1}^{N} \left( t_0 - \frac{v_i}{2} + \sqrt{\frac{v_i^2}{4} + v_i \cdot t_0} \right) = T.$$
Let us show that our formula agrees with both extreme cases mentioned above.

Indeed, for \( v_i = 0 \), we clearly have \( t_i^{\text{opt}} = t_0 \).

For \( v_i \to \infty \), we have

\[
\sqrt{\frac{v_i^2}{4} + v_i \cdot t_0} = \sqrt{\frac{v_i^2}{4} \cdot \left( 1 + \frac{4t_0}{v_i} \right)} = \frac{v_i}{2} \cdot \left( 1 + \frac{2t_0}{v_i} + o \right) = \frac{v_i}{2} + t_0 + o(1).
\]

Thus, the above expression takes the following form:

\[
t_i^{\text{opt}} = t_0 - \frac{v_i}{2} + \frac{v_i}{2} + t_0 = o(1) = 2t_0 + o(1).
\]

So, in the limit, we indeed get \( t_i^{\text{opt}} = 2t_0 \).
21. Proof that the optimal gain is always between $t_0$ and $2t_0$

- For $v_i = 0$, the above equation leads to $t_i^{\text{opt}} = t_0$.
- Thus, to prove the desired statement, it is sufficient to consider the case when $v_i > 0$.
- Let us first prove, by contradiction, that we cannot have $t_i^{\text{opt}} = 0$ for some $i$.
- Indeed, in this case, the corresponding utility $U_i$ is 0, so the product of utilities is 0.
- Thus, it cannot be the largest possible value.
- Indeed, if we simply divide $T > 0$ into $N$ equal parts, we get all utilities positive – and thus, the positive product of utilities.
22. Proof: Part 2

- Let us now prove that $t_0 > 0$.
- Due to our equation, the desired inequality is equivalent to

$$v_i + t_i^{\text{opt}} - \sqrt{v_i \cdot (v_i + t_i^{\text{opt}})} > 0.$$ 

- This is, in turn, equivalent to $v_i + t_i^{\text{opt}} > \sqrt{v_i \cdot (v_i + t_i^{\text{opt}})}$.
- Both sides are non-negative.
- For non-negative numbers, the function $x \mapsto x^2$ is strictly increasing.
- So, the last inequality is equivalent to what we will get by squaring both sides: $v_i^2 + 2v_i \cdot t_i^{\text{opt}} + (t_i^{\text{opt}})^2 > v_i^2 + v_i \cdot t_i^{\text{opt}}$.
- Subtracting the right-hand side from the left-hand side, we get the equivalent inequality $v_i \cdot t_i^{\text{opt}} + (t_i^{\text{opt}})^2 > 0$.
- This inequality is clearly true, since $v_i \geq 0$ and $t_i^{\text{opt}} > 0$.
- Thus, the original inequality $t_0 > 0$ is also true.
23. Proof: Part 3

- In our main equation, if we move $t_0$ to the left-hand side, we get an equivalent equation $L(t_i^{\text{opt}}) = 0$, where we denoted

$$L(t_i) \overset{\text{def}}{=} v_i + t_i - \sqrt{v_i \cdot (v_i + t_i)} - t_0.$$ 

- Let us prove that $L(t_i)$ is a strictly increasing function of $t_i$.

- For this purpose, it is sufficient to prove that the partial derivative of $L(t_i)$ with respect to $t_i$ is always positive.

- Here, $\frac{\partial L}{\partial t_i} = 1 - \frac{v_i}{2 \sqrt{v_i \cdot (v_i + t_i)}}$.

- Here, $v_i \cdot (v_i + t_i) \geq v_i^2$, thus $v_i \leq \sqrt{v_i \cdot (v_i + t_i)}$.

- Thus, 

$$\frac{v_i}{2 \sqrt{v_i \cdot (v_i + t_i)}} \leq \frac{1}{2} \quad \text{and} \quad \frac{\partial L}{\partial t_i} \geq 1 - \frac{1}{2} = \frac{1}{2} > 0.$$ 

- The statement is proven.
24. Proof: Part 4

- Let us now prove that for \( t_i = t_0 \), we have \( L(t_0) < 0 \) (remember that we assumed that \( v_i > 0 \)).

- Indeed, the desired inequality has the form

\[
v_i + t_0 - \sqrt{v_i \cdot (v_i + t_0)} - t_0 = v_i - \sqrt{v_i \cdot (v_i + t_0)} < 0.
\]

- This is equivalent to \( v_i < \sqrt{v_i \cdot (v_i + t_0)} \).

- Here, both sides are non-negative, so we can get an equivalent inequality by squaring both sides:

\[
v_i^2 < v_i \cdot (v_i + t_0) = v_i^2 + v_i \cdot t_0.
\]

- By subtracting \( v_i^2 \) from both sides, we get \( 0 < v_i \cdot t_0 \).

- This is clearly true since \( v_i > 0 \) and \( t_0 > 0 \).

- Thus, the equivalent inequality \( L < 0 \) is true too.
25. Proof: Part 5

- Let us now prove that for \( t_i = 2t_0 \), we have \( L(2t_0) > 0 \).
- Indeed, the desired inequality has the form
  \[
  v_i + 2t_0 - \sqrt{v_i \cdot (v_i + 2t_0)} - t_0 = v_i + t_0 - \sqrt{v_i \cdot (v_i + 2t_0)} > 0.
  \]
- This is equivalent to \( v_i + t_0 > \sqrt{v_i \cdot (v_i + 2t_0)} \).
- Here, both sides are non-negative.
- So we can get an equivalent inequality by squaring both sides:
  \[
  v_i^2 + 2v_i \cdot t_0 + t_0^2 > v_i \cdot (v_i + t_0) = v_i^2 + v_i \cdot t_0.
  \]
- By subtracting \( v_i^2 + 2v_i \cdot t_0 \) from both sides, we get \( t_0^2 > 0 \).
- This is clearly true since \( t_0 > 0 \).
- Thus, the equivalent inequality \( L > 0 \) is true too.
26. Proof: Part 6

- Now, we are ready to prove the proposition.
- Indeed, according to Part 3 of this proof, the function \( L(t_i) \) is strictly increasing.
- It is negative for \( t_i = t_0 \), it is positive for \( t_i = 2t_0 \).
- So the value \( t_i^{\text{opt}} \) for which \( L(t_i^{\text{opt}}) = 0 \) must indeed be between \( t_0 \) and \( 2t_0 \).
- The desired statement is proven.
27. Proof of the formula

- Let us move the square root to the right-hand side of the formula and $t_0$ to the left-hand side.
- Then, we get the following formula:

$$v_i + t_i^{\text{opt}} - t_0 = \sqrt{v_i \cdot (v_i + t_i^{\text{opt}})}.$$  

- Squaring both sides and opening the parentheses in the right-hand side, we get

$$v_i^2 + 2v_i \cdot t_i^{\text{opt}} - 2v_i \cdot t_0 + (t_i^{\text{opt}})^2 - 2t_i^{\text{opt}} \cdot t_0 + t_0^2 = v_i^2 + v_i \cdot t_i^{\text{opt}}.$$ 

- Let us subtract $v_i^2$ from both sides, move all the terms to the left-hand side, and combine terms proportional to $t_i^{\text{opt}}$.

- Then, we get the following quadratic equation for determining $t_i^{\text{opt}}$:

$$(t_i^{\text{opt}})^2 + t_i^{\text{opt}} \cdot (v_i - 2t_0) + (t_0^2 - 2v_i \cdot t_0) = 0.$$
28. Proof of the formula (cont-d)

- For an equation \( x^2 + p \cdot x + q = 0 \), the general solution is
  \[
  x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.
  \]

- For our equation, this leads to
  \[
  t_{\text{opt}}^i = t_0 - \frac{v_i}{2} \pm \sqrt{\left(t_0 - \frac{v_i}{2}\right)^2 - t_0^2 + 2v_i \cdot t_0}.
  \]

- The expression under the square root is equal to
  \[
  t_0^2 - v_i \cdot t_0 + \frac{v_i^2}{4} - t_0^2 + 2v_i \cdot t_0.
  \]

- The terms proportional to \( t_0^2 \) cancel each other, and the terms \(-v_i \cdot t_0\) and \(2v_i \cdot t_0\) lead to \(v_i \cdot t_0\).

- Thus, the expression under the square root is equal to
  \[
  \frac{v_i^2}{4} + v_i \cdot t_0.
  \]
29. Proof of the formula (cont-d)

- So, the last formula takes the form

\[ t_i^{\text{opt}} = t_0 - \frac{v_i}{2} \pm \sqrt{\frac{v_i^2}{4} + v_i \cdot t_0}. \]

- We have proven that \( t_i^{\text{opt}} \) is always greater than or equal to \( t_0 \).
- Thus, we cannot have the minus sign in this formula.
- So, we must have plus.
- Hence, we get the desired formula.
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