Shapley Value under Interval Uncertainty and Partial Information

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1. Practical problem: how to divide the common gain

- Often, several people – or several companies – work together and jointly gain some amount of money.
- A natural question is how to divide this amount between the participants.
- From the commonsense viewpoint, the resulting amounts should depend on how critical the participation of each person was.
- This, in turn, can be gauged by how much people would gain if some person – or several persons – did not participate.
- For example:
  - if the others could get the same amount without one of the persons participating,
  - then clearly the participation of this person did not contribute anything,
  - so this person should not get anything at all.

2. How to divide the common gain (cont-d)

- In general, to decide how to best divide the money, we need to have:
  - for each subset S of the original set of participants $N \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$,
  - some information about the largest amount $v(S)$ that they could gain if members of this set acted on their own, without involving other participants.
- Clearly, if no one does anything, we will not get any gains, i.e., $v(\emptyset) = 0$.
- Also, clearly, if we add several participants to the group, we add additional opportunities.
- So if $S \subset S'$, then we should have $v(S) \le v(S')$.
- This property is known as *monotonicity*.

3. Enter Shapley value

- The ideal case if when we know all the values $v(S)$.
- For this case, Lloyd Shapley showed that:
  - natural requirements uniquely determine the money distribution,
  - i.e., the values $x_i(v)$ for which $x_1(v) + \ldots + x_n(v) = v(N)$.
- The resulting distribution is known as the *Shapley value*.
- For this result, Professor Shapley was later awarded the Nobel Prize in Economics.
- Shapley's derivation was based on the following natural requirements.

4. First requirement: additivity

- First, Shapley considered a case when the same group of participants gets involved in two different collaboration situations, with functions

$$u(S) \text{ and } v(S).$$

- There are two ways to consider this case:
- We can consider the situations separately.
- In this case, each person i gains $x_i(u)$ in the first situation, and the value $x_i(v)$ in the second situation, to the total of $x_i(u) + x_i(v)$.
- Alternatively:
  - we can consider this case as a single situation,
  - in which each group S can gain the amount $w(S) \stackrel{\text{def}}{=} u(S) + v(S)$ if acting on their own.
- In this case, each person i gains the amount $x_i(w) = x_i(u + v)$.

5. First requirement: additivity (cont-d)

- These are the two different descriptions, but this is the same case.
- So it is reasonable to require that both descriptions lead to the exact same distribution, i.e., that we have $x_i(u+v) = x_i(u) + x_i(v)$.
- This property is known as *additivity*.

6. Second requirement: linearity

- The second natural requirement is related to the fact that the fair distribution should not depend on what monetary units we use.
- If we have a fair distribution in dollars, then we should have a fair distribution if we use Euros or any other monetary unit.
- If we use a different monetary unit that is c times smaller, then all numerical values multiply by c.
- So, instead of the original function $v(S)$, we get a new function

$$u(S) \stackrel{\text{def}}{=} c \cdot v(S).$$

- For this new function, the fair division should be the similarly re-scaled fair division for the original function $v(S)$, i.e., we should have

$$x_i(c \cdot v) = c \cdot x_i(v).$$

- This should be true for all $c > 0$.

7. Second requirement: linearity (cont-d)

- These two conditions imply that, in general, for all $c_j > 0$ and for all v_j, we have

$$x_i(c_1 \cdot v_1 + \ldots + c_m \cdot v_m) = c_1 \cdot x_i(v_1) + \ldots + c_m \cdot x_i(v_m).$$

- This property is known as *linearity*.

8. Third requirement: permutation-invariance

- The third natural requirement is that the distribution of money should not depend on how we enumerate the participants.
- In other words:
  – if we use a different ordering, i.e., if we apply a permutation

$$\pi : N \mapsto N,$$

  – then for the permuted function $u(S) \stackrel{\text{def}}{=} v(\pi(S))$, each person's gain should remain the same,
  – i.e., we should gave $x_{\pi(i)}(u) = x_i(v)$.

9. Fourth requirement: role of dummies

- The final natural requirement is that:
  - if a participant i does not bring any additional gain to any group,
  - i.e., if $v(S \cup \{i\}) = v(S)$ for all sets S,
  - then this participant should not gain anything: $x_i(v) = 0$.
- Such participants are called *dummies*.

10. Resulting formula and how to compute it

- Shapley showed that these three requirements uniquely determine the following distribution:

$$x_i(v) = \sum_{S: i \notin S} \frac{|S|! \cdot (n - |S|)!}{n!} \cdot (v(S \cup \{i\}) - v(S)).$$

- Here $|S|$ denoted the number of elements in a set S, and $n!$ is the factorial $n! \stackrel{\text{def}}{=} 1 \cdot 2 \cdot \ldots \cdot n$.
- How can we actually compute the Shapley value?
- For reasonable small n, we can compute the Shapley value explicitly, by adding 2^n terms in the above formula.
- For larger n, computing 2^n terms may not be feasible.
- E.g., for $n = 300$, this computation will take more time than the lifetime of the Universe.

11. Resulting formula and how to compute it (cont-d)

- Good news is that, as one can show:
  - the above formula is equal to
  - the expected value of the gain that the i-th participants brings when he/she enters the group in a random ordering.
- Thus, we can use reasonably fast Monte-Carlo simulation techniques for this computation.

12. Remaining problem

- In practice, we rarely know the exact values $v(S)$.
- Sometimes, we know bounds $\underline{v}(S)$ and $\overline{v}(S)$ on the values $v(S)$.
- In other words, we know the interval $[\underline{v}(S), \overline{v}(S)]$ that contains $v(S)$.
- For some sets S, we do not have any specific information about $v(S)$.
- In this case, the only thing that we can conclude about such $v(S)$ comes from the monotonicity:

$$\max(\underline{v}(T) : T \subset S) \leq v(S) \leq \min(\overline{v}(T) : S \subset T).$$

- So, the case of partial information can also be described in interval terms.
- The question is: what is the fair division in such interval case?

13. Analysis of the problem

- In this talk, we show that:
  - requirements similar to the original Shapley's ones
  - lead to a natural extension of Shapley values to such interval case.
- In the usual Shapley setting, we have a real-valued function $v(S)$ that assigns, to each subset $S \subseteq N$, a numerical value $v(S)$.
- Now, we now have a function $\mathbf{v}$ that assigns, to each set $S \subseteq N$, an interval $\mathbf{v}(S) = [\underline{v}(S), \overline{v}(S)]$.
- Based on this information, we want to assign, to each participant, a value $x_i(\mathbf{v})$.
- In the Shapley case, these assignments satisfied the requirement that the sum of values $x_i(v)$ is equal to the overall gain $v(N)$.
- In the general case of interval uncertainty, we do not know the overall gain.

14. Analysis of the problem (cont-d)

- So the only condition that we can impose is that:
  - the sum of the distributed values
  - should be within the interval of possible values of $v(N)$, i.e., that $x_1(\mathbf{v}) + \ldots + x_n(\mathbf{v}) \in \mathbf{v}(N)$.
- Of course, in case all intervals are degenerate, i.e., if $\mathbf{v}(S) = [v(S), v(S)] = \{v(S)\}$ for all S, we should get the usual Shapley value.
- Sometimes we already have the amount that we need to divide.
- This does not mean that we should restrict ourselves to such situations.
- In practice, people usually start negotiating how the money should be divided before the money actually become available.
- In this case, the amount of overall gain is usually not yet known exactly.

15. What are the natural requirements: additivity

- Similarly to the Shapley case, it is reasonable to require additivity.
- Suppose that:
  - for the situation $\mathbf{u}$, we have the interval $\mathbf{u}(S) = [\underline{u}(S), \overline{u}(S)]$ of possible gains, and
  - for the situation $\mathbf{v}$, we have the interval $\mathbf{v}(S) = [\underline{v}(S), \overline{v}(S)]$ of possible gains.
- Then overall, we can have gain values ranging from the sum of the smallest gains to the sum of the largest gains:

$$\mathbf{w}(S) = [\underline{u}(S) + \underline{v}(S), \overline{u}(S) + \overline{v}(S)].$$

- It is reasonable to require that:
  - the value $x_i(\mathbf{w})$ corresponding to this joint situation $\mathbf{w}$ be equal
  - to the sum of gains corresponding to the original situations $\mathbf{u}$ and $\mathbf{v}$, i.e., that $x_i(\mathbf{w}) = x_i(\mathbf{u}) + x_i(\mathbf{v})$.

16. Additivity (cont-d)

- The interval formed by sums of all possible pairs of numbers from two intervals is known as the *sum* of the two intervals:

$$[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] \stackrel{\text{def}}{=} [\underline{a} + \underline{b}, \overline{a} + \overline{b}].$$

- Thus, additivity can be described as

$$x_i(\mathbf{u} + \mathbf{v}) = x_i(\mathbf{u}) + x_i(\mathbf{v}).$$

17. Linearity

- Similarly, if we change the monetary unit to a new one which is c times smaller, then:
  - instead of the original interval $\mathbf{v}(S) = [\underline{v}(S), \overline{v}(S)]$,
  - we get the interval $c \cdot \mathbf{v}(S) \stackrel{\text{def}}{=} [c \cdot \underline{v}(S), c \cdot \overline{v}(S)]$.
- Thus, the requirement that the distribution not change if we simply change the monetary unit implies that

$$x_i(c \cdot \mathbf{v}) = c \cdot x_i(\mathbf{v}).$$

- Just like in the original Shapley case, this implied that for all possible values $c_j > 0$, we have

$$x_i(c_1 \cdot \mathbf{v}_1 + \ldots + c_m \cdot \mathbf{v}_m) = c_1 \cdot x_i(\mathbf{v}_1) + \ldots + c_m \cdot x_i(\mathbf{v}_m).$$

18. Distribution should not depend on the numbering of participants

- For each permutation π:
  - if we denote $\mathbf{u}(S) \stackrel{\text{def}}{=} \mathbf{v}(\pi(S))$,
  - then we should gave $x_{\pi(i)}(\mathbf{u}) = x_i(\mathbf{v})$.

19. Adding dummies

- If we add a dummy to the situation, then:
  - this dummy should get nothing, and
  - the amounts given to other participants should not change.

20. Additional requirement

- As we have mentioned earlier:
  - the distribution problem is important not only when individuals collaborate,
  - but also when companies collaborate.
- A company consists of individuals.
- So, we have two ways to looks at the situations when companies collaborate:
  - we can consider companies as participants, or
  - we can consider the companies' individuals as participants.

21. Additional requirement (cont-d)

- It is reasonable to require that: the divisions coming from these two representations should be consistent, i.e.:
  - the sum of distributions given to all company's individuals in the second approach
  - should be equal to the amount given to the company as a whole in the first approach.
- It turns out that it is sufficient to consider this additional requirement only for the simplest case, when:
  - the overall gain is the simplest interval $[0, 1]$;
  - we have only one company; and
  - to succeed, all individuals from the company must work together.
- Now, we are ready to formulate out main result.

22. Definitions

- By a *situation*, we means a pair consisting of a natural number n and of a function that assigns:
  - to each subset S of the set $N_n = \{1, \ldots, n\}$,
  - an interval $\mathbf{v}(S) = [\underline{v}(S), \overline{v}(S)]$ for which $\mathbf{v}(\emptyset) = [0, 0]$.
- By a *division scheme*, we mean a function that assigns, to each situation and to each $i \le n$, a value $x_i(\mathbf{v})$ so that
$$x_1(\mathbf{v}) + \ldots + x_n(\mathbf{v}) \in \mathbf{v}(N_n).$$
- We say that the division scheme is *additive* if for every two situations with the same n and for every i, we have
$$x_i(\mathbf{u} + \mathbf{v}) = x_i(\mathbf{u}) + x_i(\mathbf{v}).$$
- We say that the division scheme is *linear* if for every $c > 0$, we have
$$x_i(c \cdot \mathbf{v}) = c \cdot x_i(\mathbf{v}).$$

23. Definitions (cont-d)

- We say that the division scheme is *dummy-consistent* if:
  - when we form a new situation $\mathbf{u}$ by adding a participant $n+1$ to the original situation $\mathbf{v}$ so that $\mathbf{u}(S)=\mathbf{v}(S\cap N_n)$,
  - then for all $i\le n$, we have $x_i(\mathbf{u})=x_i(\mathbf{v})$, and
  - we have $x_{n+1}(\mathbf{u})=0$.
- We say that the division scheme is *permutation-invariant* if for every situation $\mathbf{v}$ and for every permutation π:
  - if we denote if we denote $\mathbf{u}(S)\stackrel{\text{def}}{=}\mathbf{v}(\pi(S))$,
  - then $x_{\pi(i)}(\mathbf{u})=x_i(\mathbf{v})$.
- Let $\mathbf{x}_n$ be a situation with n participants for which $\mathbf{v}_n(N_n)=[0,1]$ and $\mathbf{v}_n(S)=[0,0]$ for all other sets $S\subset N_n$.
- We say that the division scheme is *company-consistent* if for each n, we have $x_1(\mathbf{v}_n)+\ldots+x_n(\mathbf{v}_n)=x_1(\mathbf{v}_1)$.

24. Proposition

For each division scheme, the following two conditions are equivalent to each other:

- *the division scheme is additive, linear, dummy-consistent, permutation-invariant, and company-consistent;*
- *for some* $\alpha \in [0,1]$*, the division scheme coincides with the Shapley value for* $v(S) = \alpha \cdot \overline{v}(S) + (1-\alpha) \cdot \underline{v}(S)$*, i.e.,*

$$x_i([\underline{v}, \overline{v}]) = x_i(\alpha \cdot \overline{v} + (1-\alpha) \cdot \underline{v}).$$

25. Comment

- The above Proposition is in line with the fact – originally proven by yet another Nobelist Leo Hurwicz – that:
  - in situations when for each alternative, we only know the interval $[\underline{u}, \overline{u}]$ of possible utility values,
  - we should choose some $\alpha \in [0, 1]$ and select an alternative for which the combination $\alpha \cdot \overline{u} + (1 - \alpha) \cdot \underline{u}$ is the largest.
- Similar to that situation:
  - For $\alpha = 1$, we only take into account the best-possible values $\overline{v}(S)$ – which corresponds to perfect optimism.
  - For $\alpha = 0$, we only take into account the worst-possible values $\underline{v}(S)$ – which corresponds to perfect pessimism.
- Values α between 0 and 1 correspond to combining optimistic and pessimistic approaches.

26. In Preparation for the Proof: How Shapley Derived His Formula

- Our derivation will be similar to Shapley's.
- So let us first describe how Shapley himself came up with his formula.
- To describe a situation, we need to know the values $v(S)$ corresponding to all non-empty subsets of the original n-element set N.
- Such a set has 2^n subsets.
- So, if we exclude the empty set – for which $v(S)$ is known to be 0 – we get $2^n - 1$ nonempty sets.
- Thus, a situation can be described by $2^n - 1$ numbers.
- So, cases form a closed subset in the $(2^n - 1)$-dimensional linear space L – a set determined by the conditions that:

$$v(S) \le v(S') \text{ for all } S \subseteq S'.$$

27. How Shapley Derived His Formula (cont-d)

- In this set, we can consider:
  - for each non-empty set T,
  - a special situation v_T for which $v_T(S) = 1$ when $T \subseteq S$ and $v_T(S) = 0$ otherwise.
- There are as many such situations as non-empty subsets, i.e., $2^n - 1$.
- For this situation, distribution is easy.
- Participants who are not in T do not contribute and thus, do not get anything.
- Due to permutation-invariance, all participants from the set T get exactly the same reward, i.e., the value $1/|T|$.
- One can show that the corresponding vectors v_T form a basis in L.
- Indeed, clearly, vectors e_T for which $e_T(S) = 1$ if $S = T$ and $e_T(S) = 0$ otherwise form such a basis.

28. How Shapley Derived His Formula (cont-d)

- Let us show that each vector e_T can be obtained as a linear combination of v_T vectors.
- This will imply that vectors v_T also form a basis.
- Indeed, to get from v_T to e_T, we need to nullify the values $v(S)$ for sets $S \neq T$ for which $T \subset S$.
- First, we can nullify the values $v(T \cup \{i\})$.
- This can be done by subtracting the sum of the functions $v_{T \cup \{i\}}$, i.e., by considering the expression $v_T - \sum\limits_{i \notin T} v_{T \cup \{i\}}$.
- This subtraction nullifies the values $v(T \cup \{i\})$, but it overcorrects the values $v(T \cup \{i, j\})$: they go from 1 to -1.
- To correct these values, we add the sum of the functions $v_{T \cup \{i,j\}}$, etc.

29. How Shapley Derived His Formula (cont-d)

- As a result, we get the desired representation

$$e_T = v_T - \sum_{i \notin T} v_{T \cup \{i\}} + \sum_{i,j \notin T, i \neq j} v_{T \cup \{i,j\}} - \ldots$$

- Since vectors v_T form a basic, any vector v can be represented as a linear combination of the vectors v_T corresponding to different T:

$$v = c_1 \cdot v_{T_1} + c_2 \cdot v_{T_2} + \ldots + c_m \cdot v_{T_m}.$$

- Some of the coefficient c_i may be negative.
- We can then move the negative terms to the other side of the equality.
- Thus, we get the following equality in which all the coefficients are positive:

$$v + \sum_{k : c_k < 0} |c_k| \cdot v_{T_k} = \sum_{j : c_j > 0} c_j \cdot v_{T_j}.$$

30. How Shapley Derived His Formula (cont-d)

- Since we assumed linearity, we can conclude that

$$x_i(v) + \sum_{k:c_k<0} |c_k| \cdot x_i(v_{T_k}) = \sum_{j:c_j>0} c_j \cdot x_i(v_{T_j}).$$

- Moving the terms corresponding to $c_k < 0$ back into the right-hand side of this equality, we conclude that

$$x_i(v) = c_1 \cdot x_i(v_{T_1}) + c_2 \cdot x_i(v_{T_2}) + \ldots + c_m \cdot x_i(v_{T_m}).$$

- Since, as we have shown, the values $x_i(T)$ are uniquely determined by the above conditions, the values $x_i(v)$ are also uniquely determined.

- Thus, Shapley's result is proven.

31. Proof of Our Result

- Our result is that the division has to have the form:

$$x_i([\underline{v}, \overline{v}]) = x_i(\alpha \cdot \overline{v} + (1 - \alpha) \cdot \underline{v})$$

- It is easy to check that:
  - every division scheme of the type
  - is additive, linear, dummy-consistent, permutation-invariant, and company-consistent.
- Thus, to complete our proof, it is sufficient to prove that:
  - every additive, linear, dummy-consistent, permutation-invariant, and company-consistent division scheme
  - has the above form.
- Indeed, any interval-valued function $[\underline{v}(S), \overline{v}(S)]$ can be represented as the sum $[\underline{v}(S), \overline{v}(S)] = [\underline{v}(S), \underline{v}(S)] + [0, \Delta v(S)]$.
- Here we denoted $\Delta v(S) \stackrel{\text{def}}{=} \overline{v}(S) - \underline{v}(S)$.

32. Proof of Our Result (cont-d)

- Due to additivity, it is therefore sufficient to consider two types of situations:
  - situations of the type $[\underline{v}(S), \underline{v}(S)]$ – where intervals are actually numbers, and
  - situations of the type $[0, \Delta v(S)]$, for which the lower bound is 0.
- For the situations of the first type our requirement coincide with the requirements for the requirement that lead to the Shapley value.
- So the corresponding divisions coincide with the Shapley values $x_i(\underline{v})$.
- Let us now consider situations of the second type.
- Let α denote the value $x_1(\mathbf{v}_1)$.
- The condition that the sum of all the allocations has to be in the interval $\mathbf{v}_1(N_1)$ implies that α is located in the interval $[0, 1]$.
- For the situation $\mathbf{v}_n$, all participants are equally involved.

33. Proof of Our Result (cont-d)

- So, due to permutation-invariance, they should get equal amounts.
- Due to company-consistence, the sum of these n amounts if equal to $\mathbf{v}_1(N_1) = \alpha$.
- So each person gets the amount α/n.
- Due to dummy-consistency, for the situation $\mathbf{v}_T$ for which $\mathbf{v}_T(S) = [0, 1]$ when $T \subseteq S$ and $\mathbf{v}_T(S) = [0, 0]$ for all other S:
  - we get $x_i(\mathbf{v}_T) = \alpha/n$ for $i \in T$, and
  - we get $x_i(\mathbf{v}_T) = 0$ for all other i.
- Thus, for the situations $\mathbf{v}_T$, the corresponding division can be obtained from the Shapley division by multiplying it by α:

$$x_i(\mathbf{v}_T) = \alpha \cdot x_i(v_T).$$

- As in the original Shapley's proof, for any functions Δv, we have

$$\Delta v = c_1 \cdot v_{T_1} + c_2 \cdot v_{T_2} + \ldots + c_m \cdot v_{T_m}.$$

34. Proof of Our Result (cont-d)

- Thus, for some values c_j and for some sets T_j, we have:

$$\Delta v + \sum_{k:c_k<0} |c_k| \cdot v_{T_k} = \sum_{j:c_j>0} c_j \cdot v_{T_j}.$$

- So, we have

$$[0, \Delta v] + \sum_{k:c_k<0} |c_k| \cdot [0, v_{T_k}] = \sum_{j:c_j>0} c_j \cdot [0, v_{T_j}].$$

- Here, for each j, we have $[0, v_{T_j}] = \mathbf{v}_{T_j}$, thus

$$[0, \Delta v] + \sum_{k:c_k<0} |c_k| \cdot \mathbf{v}_{T_k} = \sum_{j:c_j>0} c_j \cdot \mathbf{v}_{T_j}.$$

- Due to additivity and linearity, we have

$$x_i([0, \Delta v]) + \sum_{k:c_k<0} |c_k| \cdot x_i(\mathbf{v}_{T_k}) = \sum_{j:c_j>0} c_j \cdot x_i(\mathbf{v}_{T_j}).$$

35. Proof of Our Result (cont-d)

- Moving the terms corresponding to $c_k < 0$ back into the right-hand side of this equality, we conclude that

$$x_i([0, \Delta v]) = c_1 \cdot x_i(\mathbf{v}_{T_1}) + c_2 \cdot x_i(\mathbf{v}_{T_2}) + \ldots + c_m \cdot x_i(\mathbf{v}_{T_m}).$$

- We know that for each j, we have $x_i(\mathbf{v}_{T_j}) = \alpha \cdot x_i(v_{T_j})$.
- Thus, we have

$$x_i([0, \Delta v]) = c_1 \cdot \alpha \cdot x_i(v_{T_1}) + c_2 \cdot \alpha \cdot x_i(v_{T_2}) + \ldots + c_m \cdot \alpha \cdot x_i(v_{T_m}) =$$
$$\alpha \cdot (c_1 \cdot x_i(v_{T_1}) + c_2 \cdot x_i(v_{T_2}) + \ldots + c_m \cdot x_i(v_{T_m})).$$

- Due to linearity of the Shapley value, the expression in the parentheses is the last part of the equality is equal to $x_i(c_1 \cdot v_{T_1} + \ldots + c_m \cdot v_{T_m})$.
- The sum in parentheses is equal to Δv, so we conclude that

$$x_i([0, \Delta v]) = \alpha \cdot x_i(\Delta v).$$

- Thus, we have

$$x_i([\underline{v}, \overline{v}]) = x_i([\underline{v}, \underline{v}]) + v([0, \overline{v} - \underline{v}]) = x_i(\underline{v}) + \alpha \cdot x_i(\overline{v} - \underline{v}).$$

36. Proof of Our Result (cont-d)

- Due to linearity of Shapley value, this implies that

$$x_i([\underline{v}, \overline{v}]) = x_i(\underline{v} + \alpha \cdot (\overline{v} - \underline{v})) = x_i(\alpha \cdot \overline{v} + (1 - \alpha) \cdot \underline{v}).$$

- The proposition is proven.

37. Comments

- The extended-to-intervals Shapley value is equal to the Shapley value of an appropriate combination of upper and lower bounds.
- Thus, we can use known methods for computing the Shapley value to compute divisions in interval-valued situations as well.
- Do we need our additional condition of company-consistency? Yes – otherwise, we could:
  - select different values α_1, α_2, etc. from the interval $[0, 1]$,
  - take $x_i(\mathbf{v}_T) = \alpha_{|T|} \cdot x_i(v_T)$, and then
  - use the above linear equation to compute $x_i(\mathbf{v})$ for all situations

$$\mathbf{v} = [\underline{v}, \overline{v}] = [\underline{v}, \underline{v}] + [0, \overline{v} - \underline{v}].$$

38. Acknowledgments

This work was supported in part by:

- National Science Foundation grants 1623190, HRD-1834620, HRD-2034030, and EAR-2225395;
- AT&T Fellowship in Information Technology;
- program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and
- a grant from the Hungarian National Research, Development and Innovation Office (NRDI).