

# Fuzzy Logic Beyond Traditional “And”-Operations

Vladik Kreinovich and Olga Kosheleva

University of Texas at El Paso  
500 W. University  
El Paso, Texas 79968, USA  
vladik@utep.edu, olgak@utep.edu

[Why do we need fuzzy...](#)

[Why do we need to go...](#)

[What are subjective...](#)

[Maximum entropy...](#)

[The corresponding...](#)

[Solution](#)

[Conclusions](#)

[Future work](#)

[Home Page](#)

[Title Page](#)



Page 1 of 41

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

## 1. Why do we need fuzzy logic in the first place?

- A large amount of human activity has been automated.
- However, in many areas, human expertise, human skills are still needed.
- We use human doctors when we are ill.
- We use human drivers and human pilots, etc.
- Not all experts and specialists are equal, some are much better than others.
- In the ideal world:
  - all diagnoses will be made by the top medical doctors,
  - all planes should be controlled by the top pilots.
- However, in reality, there are not that many top doctors, not that many top pilots, drivers, etc.

Why do we need fuzzy ...

Why do we need to go ...

What are subjective ...

Maximum entropy ...

The corresponding ...

Solution

Conclusions

Future work

Home Page

Title Page



Page 2 of 41

Go Back

Full Screen

Close

Quit

## 2. Why fuzzy logic (cont-d)

- It is not possible for them to serve all patients and all the planes.
- It is therefore desirable to use the knowledge of the top experts to help others make better decisions.
- Even, if possible, to design automatic systems that:
  - would diagnose patients, fly planes, and drive cars
  - as well as the best human specialists.
- Usually, top experts are quite willing to share their expertise, to teach others.
- But the problem is that:
  - when they share their expertise,
  - they use imprecise (“fuzzy”) words from natural language like “small”, “short”, etc.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 3 of 41

Go Back

Full Screen

Close

Quit

### 3. Why fuzzy logic (cont-d)

- This is easy to explain:
  - many of us drive cars,
  - but hardly anyone can express his/her driving experience in precise terms.
- Ask any driver how much to brake if a car 100 meters in front slows down from 100 to 95 km/h.
- A natural answer is “a little bit”.
- But an automatic system needs to know:
  - for how many milliseconds to press the brake and
  - with how many Newtons of force.
- To describe such important knowledge in precise terms, Lotfi Zadeh came up with the idea of *fuzzy logic*.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page



Page 4 of 41

Go Back

Full Screen

Close

Quit

## 4. Why fuzzy logic (cont-d)

- His main observation was that:
  - in contrast to properties like “less than 0.5 sec” which are either true or false,
  - for properties like “short” the situation is different.
- Very short time durations are absolutely short.
- Very long time durations are absolutely not short.
- However, for intermediate time durations, their “shortness” is only true to some degree.
- In a computer, “absolutely true” is usually represented by 1, and “absolutely not true” (“false”) by 0.
- It is therefore reasonable to characterize intermediate degrees of confidence by numbers between 0 and 1.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page



Page 5 of 41

Go Back

Full Screen

Close

Quit

## 5. Why fuzzy logic (cont-d)

- This is exactly what Zadeh proposed to describe properties like “small”:
  - ask the expert to indicate, for each possible value  $x$  of the corresponding quantity,
  - to what extent – on the  $[0, 1]$ -scale – this value is small.
- The resulting function  $\mu(x)$  assigns a degree to each value  $x$ .
- It is known as the *membership function* or, alternatively, as the *fuzzy set*.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀

▶

◀

▶

Page 6 of 41

Go Back

Full Screen

Close

Quit

## 6. Why we need “and”-operations (t-norms)

- Expert rules usually have several conditions.
- We can have a braking rule that describes what happens when the car is close *and* slows down a little bit.
- We can have a different rule that describes what happens when the car is close *and* slows down drastically.
- We can ask an expert, for each possible value  $d$  of the distance, to what extent this distance is close.
- We can also ask the expert:
  - for each possible value  $\Delta v$  of slowing down,
  - to what extent this value can be described as “a little bit”.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 7 of 41

Go Back

Full Screen

Close

Quit

## 7. Why we need “and”-operations (cont-d)

- But what we need, to implement this rule, is to know the degree to which:
  - for two given values  $d$  and  $\Delta v$ ,
  - to what extent  $d$  is small *and*  $\Delta v$  corresponds to “a little bit”.
- For this, we need to ask the expert’s opinion about all possible pairs of values.
- In medical diagnostics, we need to take into account the values of a dozen quantities:
  - temperature,
  - upper and lower blood pressure,
  - pulse, etc.
- Even if we use only 3 or 4 different values of each quantity, we can have  $3^{12}$  or  $4^{12}$  possible combinations.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 8 of 41

Go Back

Full Screen

Close

Quit

## 8. Why we need “and”-operations (cont-d)

- The value  $4^{12}$  is about 16 million.
- There is no way that we can ask the expert these thousands and millions of questions.
- We cannot directly ask the expert about his/her degree of confidence in all possible “and”-combinations

$$S_1 \& S_2 \& \dots \& S_n.$$

- We therefore need to be able:
  - given the expert’s degrees of confidence  $a$  and  $b$  in statement  $A$  and  $B$ ,
  - to estimate his/her degree of confidence in the composite statement  $A \& B$ .
- The value  $f_{\&}(a, b)$  of this estimate is known as the “and”-operation or, for historical reason, a  $t$ -norm.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page



Page 9 of 41

Go Back

Full Screen

Close

Quit

## 9. Why we need “and”-operations (cont-d)

- From the meaning of this operation, we can extract its natural properties.
- E.g., “ $A$  and  $B$ ” means the same as “ $B$  and  $A$ ”.
- It is therefore reasonable to require that our estimates for these two equivalent statements coincide:

$$f_{\&}(a, b) = f_{\&}(b, a).$$

- In mathematical terms, this means that the “and”-operation should be commutative.

[Why do we need fuzzy...](#)

[Why do we need to go...](#)

[What are subjective...](#)

[Maximum entropy...](#)

[The corresponding...](#)

[Solution](#)

[Conclusions](#)

[Future work](#)

[Home Page](#)

[Title Page](#)



Page 10 of 41

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

## 10. Why we need “and”-operations (cont-d)

- Similarly, “ $(A \text{ and } B) \text{ and } C$ ” means the same as “ $A \text{ and } (B \text{ and } C)$ ”.
- So, we can conclude that the resulting estimates should coincide, i.e., that we should have

$$f_{\&}(f_{\&}(a, b), c) = f_{\&}(a, f_{\&}(b, c)).$$

- In mathematical terms, this means that the “and”-operation should be associative.
- Similar arguments explain that the “and”-operation should be monotonic, continuous, etc.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page



Page 11 of 41

Go Back

Full Screen

Close

Quit

## 11. There are many possible “and”-operations

- There exist many operations that satisfy all these properties.
- We need to select the one which best reflects the expert’s reasoning.
- This selection was first done for the historically first medical expert system MYCIN.
- Since then, this has been done for many application areas.
- Interestingly, in different application areas, different “and”-operations are most adequate.

[Why do we need fuzzy...](#)

[Why do we need to go...](#)

[What are subjective...](#)

[Maximum entropy...](#)

[The corresponding...](#)

[Solution](#)

[Conclusions](#)

[Future work](#)

[Home Page](#)

[Title Page](#)

◀◀

▶▶

◀

▶

Page 12 of 41

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

## 12. Comment

- The desired most adequate “and”-operation can be determined as follows.
- For several pairs of statements  $(A_k, B_k)$ :
  - we ask the experts to estimate their degrees of confidence  $d(A_k)$ ,  $d(B_k)$ , and  $d(A_k \& B_k)$
  - in statements  $A_k$ ,  $B_k$ , and  $A_k \& B_k$ .
- Then, we find a function  $f_{\&}(a, b)$  for which

$$d(A_k \& B_k) \approx f_{\&}(d(A_k), d(B_k)) \text{ for all } k.$$

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 13 of 41

Go Back

Full Screen

Close

Quit

### 13. Why do we need to go beyond traditional “and”-operations

- So far, we have considered two extreme situations.
- To describe such situations, let us denote possible basic statements by  $S_1, \dots, S_n$ .
- In the first – ideal – situation, we know the expert’s degrees of confidence:
  - in these statements  $d(S_i)$  and
  - in all possible “and”-combinations of these statements  $d(S_{i_1} \& \dots \& S_{i_k})$ .
- The second – more realistic – situation is when we only know the degrees of confidence  $d(S_i)$  in  $S_i$ .
- In this case, we estimate our degree of confidence in each “and”-combination  $S_{i_1} \& \dots \& S_{i_k}$  as

$$f_{\&}(d(S_{i_1}), \dots, d(S_{i_k})).$$

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 14 of 41

Go Back

Full Screen

Close

Quit

## 14. Beyond traditional “and”-operations (cont-d)

- The problem is that in practice, we sometimes have intermediate situations, when:
  - we know the degrees of confidence in *some* – but not all – “and”-combinations, and
  - we are interested in estimating the expert’s degree of confidence in other “and”-combinations.
- For example:
  - in addition to the degrees of confidence  $d(S_1)$ ,  $d(S_2)$ , and  $d(S_3)$  in the three basic statements,
  - we may know the degrees of confidence in all possible pairs  $d(S_1 \& S_2)$ ,  $d(S_1 \& S_3)$ , and  $d(S_2 \& S_3)$ ,
  - and we want to estimate the degree of confidence  $d(S_1 \& S_2 \& S_3)$  in all three statements.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 15 of 41

Go Back

Full Screen

Close

Quit

## 15. Beyond traditional “and”-operations (cont-d)

- By using the traditional “and”-operation, we can get several estimates for this desired degree:

$$f_{\&}(d(S_1), d(S_2 \& S_3)), \quad f_{\&}(d(S_1 \& S_2), d(S_3)), \dots$$

- They will be, in general, different.
- Each of them takes into account some available information while ignoring other information.
- How can we:
  - take all the available information into account,
  - and thus come up with the most adequate estimate?
- We cannot do this by using the traditional “and”-operations, we need to go beyond.
- This is what we will do in this paper: we will show how such an estimate can be obtained.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 16 of 41

Go Back

Full Screen

Close

Quit

## 16. What are subjective probabilities and how they are related to fuzzy degrees

- The ultimate goal of expert's estimates is to make a decision.
- The diagnosis of a medical expert helps decide which treatment to select for a given patient.
- The decision of an expert pilot helps decide how, e.g., how to best avoid the turbulence zone.
- So, to solve problems related to expert estimates:
  - it makes sense to recall
  - how exactly these estimates are used in decision making.
- A decision maker is often uncertain about some possible events  $E$ .

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page



Page 17 of 41

Go Back

Full Screen

Close

Quit

## 17. Subjective probabilities (cont-d)

- Decision theory provides a natural scale for measuring this uncertainty.
- We compare lotteries  $L(p)$  in which a person gets \$100 with some probability  $p$  and the  $E$ -related lottery

$$L(E) \stackrel{\text{def}}{=} \text{“I get \$100 if } E, \text{ otherwise I get nothing”}.$$

- When  $p = 1$ , i.e., when the person gets \$100 unconditionally, then clearly the lottery  $L(1)$  is better.
- We will denote this situation by  $L(E) < L(1)$ .
- On the other hand, if the probability  $p$  is equal to 0, then the person does not get anything at all.
- So clearly the lottery  $L(E)$  – in which there is a change to get something – is better:  $L(0) < L(E)$ .

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀

▶

◀

▶

Page 18 of 41

Go Back

Full Screen

Close

Quit

## 18. Subjective probabilities (cont-d)

- As we increase the probability  $p$  from 0 and continue comparing, at some probability level  $p_0$ , we will switch:

from  $L(p) < L(E)$  to  $L(E) < L(p)$ .

- This threshold value  $p_0$  is known as the *subjective probability*  $ps(E)$  of the event  $E$ .
- Both degree of confidence and subjective probability describe our degree of belief that the event will happen.
- If in two situations, we have the same degree of belief, it is reasonable to expect that:
  - we have the same subjective probabilities and
  - the same degrees of confidence.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 19 of 41

Go Back

Full Screen

Close

Quit

## 19. Subjective probabilities (cont-d)

- This means that the degree of confidence uniquely determines the subjective probability.
- So,  $ps(E) = f(d(E))$  for some monotonic function  $f(d)$ .

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page



Page 20 of 41

Go Back

Full Screen

Close

Quit

## 20. How can we determine the corresponding function $f(d)$ ?

- If:
  - we know the degrees of confidence  $a$  and  $b$  in statements  $A$  and  $B$ ,
  - then we estimate the degree of confidence in  $A \& B$  as  $f_{\&}(a, b)$ .
- What if:
  - we only know the subjective probabilities  $ps(A)$  and  $ps(B)$ , and
  - we want to estimate the subjective probability  $ps(A \& B)$ ?
- In principle, we have several different probability measures with different values of  $ps(A \& B)$ .
- Which of these values should we choose?

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page



Page 21 of 41

Go Back

Full Screen

Close

Quit

## 21. How to find $f(d)$ (cont-d)

- The usual approach is to take into account that different alternatives have different uncertainty.
- This uncertainty measured, e.g., by *entropy*  $S$ .
- $S$  is the average number of “yes”-“no” questions that we need to ask to fully determine the situation.
- In general, if we have  $N$  alternatives with probabilities  $P_1, \dots, P_N$ , then entropy is equal to

$$S = - \sum_{i=1}^N P_i \cdot \log_2(P_i).$$

- For two statements  $A$  and  $B$ , we have 4 possible alternatives:  $A \& B$ ,  $A \& \neg B$ ,  $\neg A \& B$ , and  $\neg A \& \neg B$ .

[Why do we need fuzzy...](#)

[Why do we need to go...](#)

[What are subjective...](#)

[Maximum entropy...](#)

[The corresponding...](#)

[Solution](#)

[Conclusions](#)

[Future work](#)

[Home Page](#)

[Title Page](#)

◀◀

▶▶

◀

▶

Page 22 of 41

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

## 22. How to find $f(d)$ (cont-d)

- Once we know the probabilities  $p(A)$ ,  $p(B)$ , and  $P_1 = p(A \& B)$ , we can determine:

$$P_2 = p(A \& \neg B) = p(A) - p(A \& B),$$

$$P_3 = p(\neg A \& B) = p(B) - p(A \& B),$$

$$P_4 = p(\neg A \& \neg B) = 1 - p(A \& B) - p(A, \& \neg B) = p(\neg A \& B).$$

- For different values of  $p(A \& B)$ , we get, in general, different values of the entropy.
- The only thing that we know about this uncertainty is that it is in some interval  $[\underline{S}, \overline{S}]$ .
- We can guarantee that the average number of binary questions does not exceed  $\overline{S}$ .
- If we select a value  $p(A \& B)$  for which  $S < \overline{S}$ , we then artificially add certainty which is not there.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 23 of 41

Go Back

Full Screen

Close

Quit

## 23. How to find $f(d)$ (cont-d)

- We are kind of cheating by pretending that we have less uncertainty than possible.
- To avoid such cheating, it makes sense to select the value  $p(A \& B)$ :
  - for which  $S = \bar{S}$ ,
  - i.e., for which entropy is the largest possible.
- This idea is known as the *maximum entropy approach*.
- For the above case, as one can show, this approach leads to  $p(A \& B) = p(A) \cdot p(B)$ .
- In particular, for subjective probabilities, we get

$$ps(A \& B) = ps(A) \cdot ps(B).$$

- Here:  $ps(A) = f(d(A)) = f(a)$ ,  $ps(B) = f(d(B)) = f(b)$ , and  $ps(A \& B) = f(d(A \& B)) = f(f\&(a, b))$ .

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 24 of 41

Go Back

Full Screen

Close

Quit

## 24. How to find $f(d)$ (cont-d)

- So,  $f(f_{\&}(a, b)) = f(a) \cdot f(b)$ , i.e.:

$$f_{\&}(a, b) = f^{-1}(f(a) \cdot f(b)).$$

- Here  $f^{-1}(p)$  denotes the inverse function:  $f^{-1}(p)$  is the value  $d$  for which  $f(d) = p$ .
- So:
  - once we empirically determine the “and”-operation  $f_{\&}(a, b)$ ,
  - we can then determine the corresponding function  $f(d)$  as the one for which the above formula holds.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 25 of 41

Go Back

Full Screen

Close

Quit

## 25. Is not the above formula an additional restriction on possible “and”-operations?

- Can such a function  $f(d)$  be found for all possible “and”-operations?
- From the purely mathematical viewpoint, this formula is indeed a limitation.
- E.g., a popular “and”-operation  $f_{\&}(a, b) = \min(a, b)$  cannot be represented in this form.
- However, from the practical viewpoint, there is no limitation; it is known that:
  - for every “and”-operation  $f_{\&}(a, b)$  and for every  $\varepsilon > 0$ ,
  - there exists an  $\varepsilon$ -close “and”-operation of the above type.
- For sufficiently small  $\varepsilon > 0$ ,  $\varepsilon$ -close operations are practically indistinguishable.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 26 of 41

Go Back

Full Screen

Close

Quit

## 26. Maximum entropy approach is more general than using “and”-operations

- We have mentioned that the maximum entropy approach can be used:
  - to estimate the probability  $p(A \& B)$  of an “and”-statement  $A \& B$
  - when all we know are probabilities  $p(A)$  and  $p(B)$  of the basic statements.
- The same maximum entropy approach can be – and is – used in many other situations.
- It can be used to find any missing probability – including a missing probability of an “and”-combination.
- There is a natural transformation  $ps = f(d)$  from degrees of confidence  $d$  to probabilities  $ps$ .

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page



Page 27 of 41

Go Back

Full Screen

Close

Quit

## 27. Maximum entropy approach (cont-d)

- We can therefore find the missing degrees as follows:
  - first, we transform all known degrees into probabilities;
  - then, we use the Maximum Entropy approach to find the missing probabilities;
  - finally, we use the inverse function  $f^{-1}(p)$  to transform the newly found probabilities into degrees.
- Let us describe this procedure in precise terms.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page



Page 28 of 41

Go Back

Full Screen

Close

Quit

## 28. Preliminary step: Version 1

- For some application areas (and for the given class of problems):

– we have already determined the “and”-operation

$$f_{\&}(a, b)$$

– that most adequately describes the expert reasoning in this area.

- In this case, we find a function  $f(d)$  for which, for every  $a$  and  $b$ , we have  $f(f_{\&}(a, b)) = f(a) \cdot f(b)$ .

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 29 of 41

Go Back

Full Screen

Close

Quit

## 29. Preliminary step: Version 2

- In some application areas, we have not yet determined the appropriate “and”-operation.
- Then, for several pairs of statements  $(A_k, B_k)$ , we ask the experts to estimate:
  - their degrees of confidence

$$d(A_k), d(B_k), \text{ and } d(A_k \& B_k)$$

- in statements  $A_k, B_k,$  and  $A_k \& B_k.$
- Then, we find a function  $f(d)$  for which

$$f(d(A_k \& B_k)) \approx f(d(A_k)) \cdot f(d(B_k)) \text{ for all } k.$$

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page



Page 30 of 41

Go Back

Full Screen

Close

Quit

## 30. The corresponding problem

- We have several basic statements  $S_1, \dots, S_n$ ;
  - for some propositional combinations  $C_1, \dots, C_m$  of these statements,
  - we have expert estimates  $d(C_i)$  of their degree of confidence.
- We also have another propositional combination  $C$  for which we do not have the expert's estimate.
- We want to estimate the expert's degree of confidence  $d(C)$  in the statement  $C$ .

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 31 of 41

Go Back

Full Screen

Close

Quit

## 31. Example 1

- The traditional “and”-operation corresponds to the case when  $n = 2$ ,  $m = 2$ ,  $C_1 = S_1$ ,  $C_2 = S_2$  and

$$C = S_1 \& S_2.$$

- This is the case for which the traditional fuzzy “and”-operation provides a reasonable solution

$$d(C) \approx f_{\&}(d(S_1), d(S_2)) = f^{-1}(f(d(S_1), f(d(S_2)))).$$

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 32 of 41

Go Back

Full Screen

Close

Quit

## 32. Example 2

- Here is an example when we need to go beyond the traditional “and”-operation:  $n = 3$ ,  $m = 6$ ,

$$C_1 = S_1, \quad C_2 = S_2, \quad C_3 = S_3,$$

$$C_4 = S_1 \& S_2, \quad C_5 = S_1 \& S_3, \quad C_6 = S_2 \& S_3,$$

$$C = S_1 \& S_2 \& S_3.$$

- We know the values  $d_i \stackrel{\text{def}}{=} d(S_i)$  and  $d_{ij} \stackrel{\text{def}}{=} d(S_i \& S_j)$ .
- We want to estimate the degree  $d \stackrel{\text{def}}{=} d(S_1 \& S_2 \& S_3)$ .

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page



Page 33 of 41

Go Back

Full Screen

Close

Quit

### 33. Solution

- First, we transform all known degrees  $d(C_i)$  into subjective probabilities, by computing  $ps(C_i) = f(d(C_i))$ .
- Then:
  - among all the probability distributions with given values  $ps(C_i)$ ,
  - we find the one for which the entropy is the largest possible.
- For this maximum-entropy distribution, we determine the (subjective) probability  $ps(C)$ .
- Finally, we transform this probability back into degrees by computing  $d(C) = f^{-1}(ps(C))$ .

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 34 of 41

Go Back

Full Screen

Close

Quit

## 34. Comment

- For  $n$  statements, to get a full probability distribution:
  - we need to know the probability of all  $2^n$  atomic combinations, i.e.,
  - combinations of the form  $S_1^{\varepsilon_1} \& \dots \& S_n^{\varepsilon_n}$ , where  $\varepsilon_i$  is either  $+$  or  $-$ ,  $S_i^+$  means  $S_i$ , and  $S_i^-$  means  $\neg S_i$ .
- Thus, the entropy is

$$- \sum_{\varepsilon_1, \dots, \varepsilon_n} ps(S_1^{\varepsilon_1} \& \dots \& S_n^{\varepsilon_n}) \cdot \log_2(ps(S_1^{\varepsilon_1} \& \dots \& S_n^{\varepsilon_n})).$$

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 35 of 41

Go Back

Full Screen

Close

Quit

## 35. Example

- In the second example, first, we compute the probabilities  $p_i = ps(S_i) = f(d_i)$  and  $p_{ij} = ps(S_i \& S_j) = f(d_{ij})$ .
- Suppose that we know the subjective probability  $p$  of the desired statement  $S_1 \& S_2 \& S_3$ .
- Then, we can then determine the (subjective) probabilities of all 8 atomic statements:

$$ps(S_1 \& S_2 \& S_3) = p; \quad ps(S_1 \& S_2 \& \neg S_3) = p_{12} - p;$$

$$ps(S_1 \& \neg S_2 \& S_3) = p_{13} - p; \quad ps(\neg S_1 \& S_2 \& S_3) = p_{23} - p;$$

$$ps(S_1 \& \neg S_2 \& \neg S_3) = p_1 - p_{12} - p_{13} + p;$$

$$ps(\neg S_1 \& S_2 \& \neg S_3) = p_2 - p_{12} - p_{23} + p;$$

$$ps(\neg S_1 \& \neg S_2 \& S_3) = p_3 - p_{13} - p_{23} + p;$$

$$ps(\neg S_1 \& \neg S_2 \& \neg S_3) = 1 - p_1 - p_2 - p_3 + p_{12} + p_{13} + p_{23} - p.$$

- The value  $p$  can be determined by maximizing the corresponding entropy.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀

▶

◀

▶

Page 36 of 41

Go Back

Full Screen

Close

Quit

## 36. Comment

- The simplified case is when  $p_1 = p_2 = p_3$  and

$$p_{12} = p_{13} = p_{23}.$$

- In this case, the expression for the entropy has the form

$$\begin{aligned} & -p \cdot \log_2(p) - 3(p_{ij} - p) \cdot \log_2(p_{ij} - p) - \\ & 3(p_i - 2p_{ij} + p) \cdot \log_2(p_i - 2p_{ij} + p) - \\ & (1 - 3p + 3p_{ij} - p) \cdot \log_2(1 - 3p + 3p_{ij} - p). \end{aligned}$$

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page



Page 37 of 41

Go Back

Full Screen

Close

Quit

## 37. Conclusions

- Traditional fuzzy logic allow us:
  - based on the expert's degrees of certainty in several statements  $S_1$  & ... &  $S_n$ ,
  - to estimate the expert's degree of certainty in different propositional combinations of  $S_i$ .
- Often, however:
  - in addition to expert's degrees of confidence in the basic statements  $S_i$ ,
  - we also know expert's degrees of confidence in several propositional combinations of  $S_i$ .

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 38 of 41

Go Back

Full Screen

Close

Quit

## 38. Conclusions (cont-d)

- It is desirable:
  - to take this additional information into account
  - when estimating the expert's degree of certainty in the desired combination  $C$ .
- In this talk, we provide an algorithm:
  - that solves this problem for the case
  - when we only consider “and”-operations.

[Why do we need fuzzy...](#)

[Why do we need to go...](#)

[What are subjective...](#)

[Maximum entropy...](#)

[The corresponding...](#)

[Solution](#)

[Conclusions](#)

[Future work](#)

[Home Page](#)

[Title Page](#)

◀◀

▶▶

◀

▶

Page 39 of 41

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)

## 39. Future work

- In this talk, we only considered “and”-combinations of the original statements.
- It is desirable to extend our result to general propositional combinations, when:
  - in addition to “and”,
  - we also use negation and “or”-combinations.

Why do we need fuzzy...

Why do we need to go...

What are subjective...

Maximum entropy...

The corresponding...

Solution

Conclusions

Future work

Home Page

Title Page



Page 40 of 41

Go Back

Full Screen

Close

Quit

## 40. Acknowledgments

This work was supported in part by the National Science Foundation grants:

- 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science);
- HRD-1834620 and HRD-2034030 (CAHSI Includes).

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478.

[Why do we need fuzzy...](#)

[Why do we need to go...](#)

[What are subjective...](#)

[Maximum entropy...](#)

[The corresponding...](#)

[Solution](#)

[Conclusions](#)

[Future work](#)

[Home Page](#)

[Title Page](#)



Page 41 of 41

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)