If Many Physicists Are Right and No Physical Theory Is Perfect, Then by Using Physical Observations, We Can Feasibly Solve Almost All Instances of Each NP-Complete Problem

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1. Outline

- Many real-life problems are, in general, NP-complete.
- Informally speaking, these problems are difficult to solve on computers based on the usual physical techniques.
- A natural question is: can the use of non-standard physics speed up the solution of these problems?
- This question has been analyzed, e.g.:
  - for quantum field theory,
  - for cosmological solutions with wormholes and/or casual anomalies.
- However, many physicists believe that no physical theory is perfect; in this talk, we show that:
  - if such a no-perfect-theory principle is true,
  - then we can feasibly solve almost all instances of each NP-complete problem.

2. Solving NP-Complete Problems Is Important

- In practice, we often need to find a solution that satisfies a given set of constraints.
- At a minimum, we need to check whether such a solution is possible.
- Once we have a candidate, we can feasibly check whether this candidate satisfies all the constraints.
- In theoretical computer science, “feasibly” is usually interpreted as computable in polynomial time.
- The class of all such problems is called NP.
- Example: satisfiability – checking whether a formula like $(v_1 \vee \neg v_2 \vee v_3) \,\&\, (v_4 \vee \neg v_2 \vee \neg v_5) \,\&\, \ldots$ can be true.
- Each problem from the class NP can be algorithmically solved by trying all possible candidates.

3. NP-Complete Problems (cont-d)

- For example, we can try all 2^n possible combinations of true-or-false values $v_1, \ldots, v_n$.
- For medium-size inputs, e.g., for $n \approx 300$, the resulting time 2^n is larger than the lifetime of the Universe.
- So, these exhaustive search algorithms are not practically feasible.
- It is not known whether problems from the class NP can be solved feasibly (i.e., in polynomial time).
- This is the famous open problem $P \stackrel{?}{=} NP$.
- We know that some problems are *NP-complete*: every problem from NP can be reduced to it.
- So, it is very important to be able to efficiently solve even one NP-hard problem.

4. Can Non-Standard Physics Speed Up the Solution of NP-Complete Problems?

- NP-complete means difficult to solve on computers based on the usual physical techniques.
- A natural question is: can the use of non-standard physics speed up the solution of these problems?
- This question has been analyzed for several specific physical theories, e.g.:
  - for quantum field theory,
  - for cosmological solutions with wormholes and/or casual anomalies.
- So, a scheme based on a theory may not work.

5. No Physical Theory Is Perfect

- If a speed-up is possible within a given theory, is this a satisfactory answer?
- In the history of physics,
  - always new observations appear
  - which are not fully consistent with the original theory.
- For example, Newton's physics was replaced by quantum and relativistic theories.
- Many physicists believe that every physical theory is approximate.
- For each theory T, inevitably new observations will surface which require a modification of T.
- Let us analyze how this idea affects computations.

6. No Physical Theory Is Perfect: How to Formalize This Idea

- *Statement:* for every theory, eventually there will be observations which violate this theory.
- To formalize this statement, we need to formalize what are *observations* and what is a *theory*.
- Most sensors already produce *observation* in the computer-readable form, as a sequence of 0s and 1s.
- Let ω_i be the bit result of an experiment whose description is i.
- Thus, all past and future observations form a (potentially) infinite sequence $\omega = \omega_1\omega_2\ldots$ of 0s and 1s.
- A physical *theory* may be very complex.
- All we care about is which sequences of observations ω are consistent with this theory and which are not.

7. What Is a Physical Theory?

- So, a physical theory T can be defined as the set of all sequences ω which are consistent with this theory.
- A physical theory must have at least one possible sequence of observations: $T \neq \emptyset$.
- A theory must be described by a finite sequence of symbols: the set T must be *definable.*
- How can we check that an infinite sequence $\omega = \omega_1\omega_2\ldots$ is consistent with the theory?
- The only way is check that for every n, the sequence $\omega_1\ldots\omega_n$ is consistent with T; so:

$$\forall n\, \exists \omega^{(n)} \in T\, (\omega^{(n)}_1 \ldots \omega^{(n)}_n = \omega_1 \ldots \omega_n) \Rightarrow \omega \in T.$$

- In mathematical terms, this means that T is *closed* in the Baire metric $d(\omega, \omega') \stackrel{\text{def}}{=} 2^{-N(\omega,\omega')}$, where

$$N(\omega, \omega') \stackrel{\text{def}}{=} \max\{k : \omega_1 \ldots \omega_k = \omega'_1 \ldots \omega'_k\}.$$

8. What Is a Physical Theory: Definition

- A theory must predict something new.
- So, for every sequence $\omega_1 \ldots \omega_n$ consistent with T, there is a continuation which does not belong to T.
- In mathematical terms, T is *nowhere dense.*
- *By a* physical theory, *we mean a non-empty closed nowhere dense definable set T.*
- *A sequence ω is* consistent with the no-perfect-theory principle *if it does not belong to any physical theory.*
- In precise terms, ω does not belong to the union of all definable closed nowhere dense set.
- There are countably many definable set, so this union is *meager* (= *Baire first category*).
- Thus, due to Baire Theorem, such sequences ω exist.

9. How to Represent Instances of an NP-Complete Problem

- For each NP-complete problem $\mathcal{P}$, its instances are sequences of symbols.
- In the computer, each such sequence is represented as a sequence of 0s and 1s.
- We can append 1 in front and interpret this sequence as a binary code of a natural number i.
- In principle, not all natural numbers i correspond to instances of a problem $\mathcal{P}$.
- We will denote the set of all natural numbers which correspond to such instances by $S_{\mathcal{P}}$.
- For each $i \in S_{\mathcal{P}}$, we denote the correct answer (true or false) to the i-th instance of the problem $\mathcal{P}$ by $s_{\mathcal{P},i}$.

10. What We Mean by Using Physical Observations in Computations

- In addition to performing computations, our computational device can:
  - produce a scheme i for an experiment, and then
  - use the result ω_i of this experiment in future computations.
- In other words, given an integer i, we can produce ω_i.
- In precise terms, the use of physical observations in computations means that use ω as an *oracle*.

11. Main Result

- A *ph-algorithm* $\mathcal{A}$ is an algorithm that uses an oracle ω consistent with the no-perfect-theory principle.
- The result of applying an algorithm $\mathcal{A}$ using ω to an input i will be denoted by $\mathcal{A}(\omega, i)$.
- We say that a feasible ph-algorithm $\mathcal{A}$ *solves almost all instances of an NP-complete problem* $\mathcal{P}$ if:

$$\forall \varepsilon_{>0}\, \forall n\, \exists N_{\geq n} \left(\frac{\#\{i \leq N : i \in S_{\mathcal{P}}\, \&\, \mathcal{A}(\omega, i) = s_{\mathcal{P},i}\}}{\#\{i \leq N : i \in S_{\mathcal{P}}\}} > 1 - \varepsilon \right).$$

- Restriction to sufficiently long inputs $N \geq n$ makes sense: for short inputs, we can do exhaustive search.
- **Theorem.** *For every NP-complete problem* $\mathcal{P}$*, there is a feasible ph-alg.* $\mathcal{A}$ *solving almost all instances of* $\mathcal{P}$*.*

12. This Result Is the Best Possible

- Our result is the best possible, in the sense that the use of physical observations cannot solve *all* instances:
- **Proposition.** *If P≠NP, then no feasible ph-algorithm $\mathcal{A}$ can solve all instances of $\mathcal{P}$.*
- Can we prove the result for *all* N starting with some N_0?
- We say that a feasible ph-algorithm $\mathcal{A}$ *δ-solves* $\mathcal{P}$ if

$$\exists N_0\, \forall N \geq N_0 \left(\frac{\#\{i \leq N : i \in S_{\mathcal{P}}\, \&\, \mathcal{A}(\omega, i) = s_{\mathcal{P},i}\}}{\#\{i \leq N : i \in S_{\mathcal{P}}\}} > \delta \right).$$

- **Proposition.** *For every NP-complete problem $\mathcal{P}$ and for every $\delta > 0$:*
  - *if there exists a feasible ph-algorithm $\mathcal{A}$ that δ-solves $\mathcal{P}$,*
  - *then there is a feasible algorithm $\mathcal{A}'$ that also δ-solves $\mathcal{P}$.*

13. Proof of the Main Result

- As $\mathcal{A}$, given an instance i, we simply produce the result ω_i of the i-th experiment.
- Let us prove, by contradiction, that for every $\varepsilon > 0$ and for every n, there exists an integer $N \geq n$ for which

$$\#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, \omega_i = s_{\mathcal{P},i}\} > (1-\varepsilon)\cdot\#\{i \leq N : i \in S_{\mathcal{P}}\}.$$

- The assumption that this property is not satisfied means that for some $\varepsilon > 0$ and for some integer n, we have

$$\forall N_{\geq n}\, \#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, \omega_i = s_{\mathcal{P},i}\} \leq (1-\varepsilon)\cdot\#\{i \leq N : i \in S_{\mathcal{P}}\}.$$

- Let $T \stackrel{\text{def}}{=} \{x : \#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \leq$

$$(1-\varepsilon)\cdot\#\{i \leq N : i \in S_{\mathcal{P}}\} \text{ for all } N \geq n\}.$$

- We will prove that this set T is a physical theory (in the sense of the above definition); then $\omega \notin T$.

14. Proof (cont-d)

- *Reminder:* $T = \{x : \#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \le$

$$(1-\varepsilon)\cdot \#\{i \le N : i \in S_{\mathcal{P}}\} \text{ for all } N \ge n\}.$$

- By definition, a physical theory is a set which is non-empty, closed, nowhere dense, and definable.
- Non-emptiness is easy: the sequence $x_i = \neg s_{\mathcal{P},i}$ for $i \in S_{\mathcal{P}}$ belongs to T.
- One can prove that T is closed, i.e., if $x^{(m)} \in T$ for which $x^{(m)} \to \omega$, then $x \in T$.
- Nowhere dense means that for every finite sequence $x_1 \ldots x_m$, there exists a continuation $x \notin T$.
- Indeed, for extension, we can take $x_i = s_{\mathcal{P},i}$ if $i \in S_{\mathcal{P}}$.
- Finally, we have an explicit definition of T, so T is definable.

15. Proof of First Proposition

- Let us assume that P≠NP; we want to prove that for every feasible ph-algorithm $\mathcal{A}$, it is not possible to have

$$\forall N\,(\#\{i \le N : i \in S_{\mathcal{P}}\,\&\,\mathcal{A}(\omega, i) = s_{\mathcal{P},i}\} = \#\{i \le N : i \in S_{\mathcal{P}}\}).$$

- Let us consider, for each feasible ph-algorithm $\mathcal{A}$,

$$T(\mathcal{A}) \stackrel{\text{def}}{=} \{x : \#\{i \le N : i \in S_{\mathcal{P}}\,\&\,\mathcal{A}(x, i) = s_{\mathcal{P},i}\} = \#\{i \le N : i \in S_{\mathcal{P}}\} \text{ for all } N\}.$$

- Similarly to the proof of the main result, we can show that this set $T(\mathcal{A})$ is closed and definable.
- To prove that $T(\mathcal{A})$ is nowhere dense, we extend $x_1 \ldots x_m$ by 0s; then $x \in T$ would mean P=NP.
- If $T(\mathcal{A}) \ne \emptyset$, then $T(\mathcal{A})$ is a theory, so $\omega \notin T(\mathcal{A})$.
- If $T(\mathcal{A}) = \emptyset$, this also means that $\mathcal{A}$ does not solve all instances of the problem $\mathcal{P}$ – no matter what ω we use.

16. Proof of Second Proposition

- Let us assume that no non-oracle feasible algorithm δ-solves the problem $\mathcal{P}$.
- Let's consider, for each N_0 and feasible ph-alg. $\mathcal{A}$,

$$T(\mathcal{A}, N_0) \stackrel{\text{def}}{=} \{x : \#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(x,i) = s_{\mathcal{P},i}\} > \delta \cdot \#\{i \le N : i \in S_{\mathcal{P}}\} \text{ for all } N \ge N_0\}.$$

- We want to prove that $\forall N_0\, (\omega \notin T(\mathcal{A}, N_0))$.
- Similarly to the proof of the Main Result, we can show that $T(\mathcal{A}, N_0)$ is closed and definable.
- To prove that $T(\mathcal{A}, N_0)$ is nowhere dense, we extend $x_1 \ldots x_m$ by 0s.
- If $T(\mathcal{A}, N_0) \ne \emptyset$, then $T(\mathcal{A}, N_0)$ is a theory hence $\omega \notin T(\mathcal{A}, N_0)$.
- If $T(\mathcal{A}, N_0) = \emptyset$, then also $\omega \notin T(\mathcal{A}, N_0)$.

17. Appendix: A Formal Definition of Definable Sets

- Let $\mathcal{L}$ be a theory.
- Let $P(x)$ be a formula from $\mathcal{L}$ for which the set $\{x \mid P(x)\}$ exists.
- We will then call the set $\{x \mid P(x)\}$ *$\mathcal{L}$-definable.*
- Crudely speaking, a set is $\mathcal{L}$-definable if we can explicitly *define* it in $\mathcal{L}$.
- All usual sets are definable: $\mathbb{N}$, $\mathbb{R}$, etc.
- Not every set is $\mathcal{L}$-definable:
  - every $\mathcal{L}$-definable set is uniquely determined by a text $P(x)$ in the language of set theory;
  - there are only countably many texts and therefore, there are only countably many $\mathcal{L}$-definable sets;
  - so, some sets of natural numbers are not definable.

18. How to Prove Results About Definable Sets

- Our objective is to be able to make mathematical statements about $\mathcal{L}$-definable sets. Therefore:
  - in addition to the theory $\mathcal{L}$,
  - we must have a stronger theory $\mathcal{M}$ in which the class of all $\mathcal{L}$-definable sets is a countable set.
- For every formula F from the theory $\mathcal{L}$, we denote its Gödel number by $\lfloor F \rfloor$.
- We say that a theory $\mathcal{M}$ is *stronger* than $\mathcal{L}$ if:
  - $\mathcal{M}$ contains all formulas, all axioms, and all deduction rules from $\mathcal{L}$, and
  - $\mathcal{M}$ contains a predicate $\mathrm{def}(n, x)$ such that for every formula $P(x)$ from $\mathcal{L}$ with one free variable,

$$\mathcal{M} \vdash \forall y\, (\mathrm{def}(\lfloor P(x) \rfloor, y) \leftrightarrow P(y)).$$

19. Existence of a Stronger Theory

- As $\mathcal{M}$, we take $\mathcal{L}$ plus all above equivalence formulas.
- Is $\mathcal{M}$ consistent?
- Due to compactness, we prove that for any $P_1(x), \ldots, P_m(x)$, $\mathcal{L}$ is consistent with the equivalences corr. to $P_i(x)$.
- Indeed, we can take

$$\mathrm{def}(n, y) \leftrightarrow (n = \lfloor P_1(x) \rfloor \,\&\, P_1(y)) \vee \ldots \vee (n = \lfloor P_m(x) \rfloor \,\&\, P_m(y)).$$

- This formula is definable in $\mathcal{L}$ and satisfies all m equivalence properties.
- Thus, the existence of a stronger theory is proven.
- The notion of an $\mathcal{L}$-definable set can be expressed in $\mathcal{M}$: S is $\mathcal{L}$-definable iff $\exists n \in \mathbb{N}\, \forall y\, (\mathrm{def}(n, y) \leftrightarrow y \in S)$.
- So, all statements involving definability become statements from the $\mathcal{M}$ itself, *not* from metalanguage.