Why Feynman Path Integration?

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1. Why Use Quantum Effects in Computing?

- *Fact:* computers are fast.
- *Challenge:* computer are not yet fast enough:
  - to predict weather more accurately and earlier,
  - to run industrial robots more efficiently,
  - to power complex simulations of forest fires,

  and for many other applications, we need higher computing speed.
- *How:* to increase the computing speed, we must make signals travel faster between computer components.
- *Limitation:* signals cannot travel faster than the speed of light, and they already travel at about that speed.
- *Conclusion:* the size of computer components must be reduced.

2. Why Quantum Effects in Computing? (cont-d)

- *Reminder:* the size of computer components must be reduced.
- *Fact:* the sizes of these components have almost reached the sizes of atoms and molecules.
- *Fact:* these molecules follow the rules of quantum mechanics.
- *Conclusion:* quantum computing is necessary.
- *Resulting question:* how to describe these quantum effects?

3. Need for Quantization

- Since the early 1900s, we know that we need to take into account quantum effects. Thus:
  - for every non-quantum physical theory describing a certain phenomenon, be it
    * mechanics
    * or electrodynamics
    * or gravitation theory,
  - we must come up with an appropriate quantum theory.
- *Traditional quantization methods:* replace the scalar physical quantities with the corresponding operators.
- *Problem:* operators are non-commutative, $px \neq xp$.
- *Conclusion:* several different quantum versions of each classical theory.

4. Towards Feynman's Approach: Least Action Principle

- *Laws of physics* have been traditionally described in terms of differential equations.
- For *fundamental* physical phenomena, not all differential equations make sense.
- *Example:* we need conservation of fundamental physical quantities (energy, momentum, etc.)
- It turns out that
  - all known fundamental physical equations can be described in terms of minimization, and
  - in general, equations following from a minimization principle lead to conservation laws.

5. Towards Feynman's Approach: Least Action Principle (cont-d)

- *Idea:* we can assign, to each trajectory $\gamma(t)$, we can assign a value $S(\gamma)$ such that
  - among all possible trajectories,
  - the actual one is the one for which the value $S(\gamma)$ is the smallest possible.
- This value $S(\gamma)$ is called *action*.
- The principle that action is minimized along the actual trajectory is called the *minimal action principle*.
- *Feynman's idea:* the probability to get from the state $\underline{\gamma}$ to the state $\overline{\gamma}$ is proportional to $|\psi(\underline{\gamma} \to \overline{\gamma})|^2$, where

$$\psi = \sum_{\gamma:\underline{\gamma}\to\overline{\gamma}} \exp\left(\mathrm{i} \cdot \frac{S(\gamma)}{\hbar}\right).$$

6. Feynman's Approach: Successes and Challenges

- *Successes:* Feynman's approach is an efficient computing tool:
  - we can expand the corresponding expression, and
  - represent the resulting probability as a sum of an infinite series.
- Each term of this series can be described by an appropriate graph called *Feynman diagram.*
- *Foundational challenge:* why the above formula?
- *What we do in this talk:* we provide a natural explanation for Feynman's path integration formula.

7. First Idea: An Alternative Representation of the Original Theory

- *Reminder:* a physical theory is a functional S that assigns, to every path γ, the value of the action $S(\gamma)$.
- From this viewpoint, a priori, all the paths are equivalent, they only differ by the corresponding values $S(\gamma)$.
- In other words, what is important is the frequency with which we encounter different values $S(\gamma)$:
  - if among N paths, only one has this value of the action, this frequency is $1/N$,
  - if two, the frequency is $2/N$, etc.
- In mathematical terms, this means that we consider the action $S(\gamma)$ as a *random variable.*
- One possible way to describe a random variable α is by its characteristic function $\chi_\alpha(\omega) \stackrel{\text{def}}{=} E[\exp(\mathrm{i}\cdot\omega\cdot\alpha)]$.

8. An Alternative Representation of the Original Theory (cont-d)

- *Reminder:* we consider $\alpha = S(\gamma)$ as a random variable.
- *Reminder:* we describe α via its characteristics function $\chi_\alpha(\omega) \stackrel{\text{def}}{=} E[\exp(\mathrm{i} \cdot \omega \cdot \alpha)]$.
- *Conclusion:*

$$\chi(\omega) = \frac{1}{N} \cdot \sum_{\gamma} \exp(\mathrm{i} \cdot S(\gamma) \cdot \omega).$$

- *Reminder:* Feynman's formula

$$\psi = \sum_{\gamma : \underline{\gamma} \to \overline{\gamma}} \exp\left(\mathrm{i} \cdot \frac{S(\gamma)}{\hbar}\right).$$

- *Observation:* Feynman's formula is $\chi(1/\hbar)$.
- *Comment:* this is not yet a derivation, since there are many ways to represent a random variable.

9. Second Idea: Appropriate Behavior for Independent Physical Systems

- *Objective:* derive a formula that transforms a functional $S(\gamma)$ into transition probabilities.
- *Typical situation:* the physical system consists of two subsystems.
- In this case, each state γ of the composite system is a pair $\gamma = (\gamma_1, \gamma_2)$ consisting of
  - a state γ_1 of the first subsystem and
  - the state γ_2 of the second subsystem.
- Often, these subsystems are *independent.*
- Due to this independence,

$$P((\gamma_1, \gamma_2) \to (\gamma_1', \gamma_2')) = P_1(\gamma_1 \to \gamma_1') \cdot P_2(\gamma_2 \to \gamma_2').$$

10. Independent Physical Systems (cont-d)

- *Reminder:*

$$P((\gamma_1,\gamma_2)\to(\gamma_1',\gamma_2'))=P_1(\gamma_1\to\gamma_1')\cdot P_2(\gamma_2\to\gamma_2').$$

- In physics, independence is usually described as

$$S((\gamma_1,\gamma_2))=S_1(\gamma_1)+S_2(\gamma_2).$$

- In probabilistic terms, this means that we have the sum of two independent random variables. So:
  - the probability corresponding to the sum of independent random variables
  - is equal to the product of corresponding probabilities.

- *Fact:* for the sum, characteristic functions multiply:

$$\chi_{\alpha_1+\alpha_2}(\omega)=\chi_{\alpha_1}(\omega)\cdot\chi_{\alpha_2}(\omega).$$

11. Independent Physical Systems (cont-d)

- *Reminder:* we have $p(\chi_1 \cdot \chi_2) = p(\chi_1) \cdot p(\chi_2)$, i.e., to

$$p(\chi_1(\omega_1) \cdot \chi_2(\omega_1), \ldots, \chi_1(\omega_n) \cdot \chi_2(\omega_n), \ldots) =$$
$$p(\chi_1(\omega_1), \ldots, \chi_1(\omega_n), \ldots) \cdot p(\chi_2(\omega_1), \ldots, \chi_2(\omega_n), \ldots).$$

- *To simplify:* use log-log scale:
  - $P \stackrel{\text{def}}{=} \ln(p)$ as the new dependent variable, and
  - the values $Z_i = X_i + \mathrm{i} \cdot Y_i \stackrel{\text{def}}{=} \ln(\chi(\omega_i))$ as the new independent variables:

$$P(Z_1, \ldots, Z_n, \ldots) = \ln p(\exp(Z_1), \ldots, \exp(Z_n), \ldots)$$

- Then, $P(Z + Z') = P(Z) + P(Z')$, so P is linear:

$$P(Z) = \sum_i (a(\omega_i) \cdot X(\omega_i) + b(\omega_i) \cdot Y(\omega_i)).$$

- Thus, $p(\chi) = \exp(P(\ln(\chi)) = \prod_i |\chi(\omega_i)|^{a_i}$.

12. Third Idea: Maximal Set of Possible Future States

- *Reminder:* $p(\chi) = \prod_i |\chi(\omega_i)|^{a_i}$.
- *Reminder:* each value $\chi(\omega_i)$ is equal to the Feynman sum, with $\hbar_i = 1/\omega_i$.
- *Question:* why only one such term?
- *In classical physics:* once we know the initial state γ, we can uniquely predict all future states γ'.
- *In quantum physics:* we can only predict probabilities.
- *Sometimes:* $\psi(x) = 0$, so the state x is not possible.
- *Idea:* select a theory for which the set I of inaccessible states γ' is the smallest possible.
- *Fact:* a state is inaccessible if $\chi(\omega_i) = 0$ for some i.
- *Conclusion:* the set I is the smallest when we have only one such term: $p(\chi) = |\chi(\omega)|^a$.

13. Fourth Idea: Analyticity and Simplicity

- *Reminder:* $p(\chi) = |\chi(\omega)|^a$ for some a.
- *In physics:* most dependencies are real-analytical, i.e., expandable in convergent Taylor series.
- *For* $z = x + \mathrm{i} \cdot y$*:* we have

$$|z|^a = \left(\sqrt{x^2 + y^2}\right)^a = \left(x^2 + y^2\right)^{a/2}.$$

- *This expression* is analytical at $z = 0$ if and only if a is an even natural number ($a = 0, 2, 4, \ldots$)
- *Fact:* case $a = 0$ is trivial: all transition probabilities are the same.
- *Conclusion:* the simplest non-trivial case is $a = 2$.
- *Thus:* we have indeed justified Feynman integration.

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