An Analysis of Transition-to-Proof Course Students' Proving Difficulties

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- We analyzed students' proving difficulties on a final examination to collect information for teaching and redesigning our inquiry-based transition-to-proof course.
- The proving difficulties we examined are mostly about the *proving process,* not mathematical content.

Theoretical Perspective

- We view a proof construction as a *sequence of mental or physical actions.*

- These actions arise from a person's (inner interpretation of) situations, in a partly completed proof construction.

- When similar situations are followed by similar actions, an “automated link” may be learned between such situations and actions. (Bargh, pp. 30-37)
- The situation is then followed by the action, without the need for any conscious processing between the two.

Example of an Action

- In a situation calling for *C* to be proved from *A* or *B*, one constructs 2 independent subproofs arriving at *C*, one supposing *A*, the other supposing *B*.

- The action in this case is setting up the proof this way.

- If one has had repeated experience with such proofs, one does not have to think about doing or justifying this action, one just does it.

Observations

- *Automating actions* reduces the burden on working memory, which is a limited resource.
- *Some actions are beneficial* and should be initiated or encouraged.
- *Others are detrimental* and should be eliminated or discouraged.

Two Concepts We Have Found Useful

Proof framework

- First level: Write the top and bottom parts of a proof that come and just from the theorem statement.
- Second level: Unpack the conclusion and do the “same thing”.

Exploration (doing something of unknown value, e.g., finding/constructing objects, manipulating them)

- During proof construction, the partly completed proof and scratchwork can be used as aids to reflection and to reduce the burden on working memory.
- A proof is the result of some of the actions in a proof construction.
- As students are learning proof construction, many actions, such as the construction of a proof framework, can be automated.
- A good way to learn them is through “coached experience” (like riding a bicycle or playing soccer).

The Course

- The *inquiry-based course*, from which the data came, was taught entirely from notes with students constructing original proofs, presenting them in class, and receiving critiques.
- In order to coordinate with later courses, the notes included some theorems about sets, functions, real analysis, and abstract algebra.
- Logic was taught in context as the need arose, mainly through the discussion of students' logical errors.
- The examination questions all asked for original proofs of theorems that weren't in the course notes, but used definitions and theorems from the course notes, which were available during the examination.

Data Analysis

- We analyzed all 16 take-home and all 16 in-class final examination papers from the course.
- These were analyzed through several iterations, looking for *categories of students' proving difficulties,* in particular, *actions taken or not taken*, until the researchers came to an agreement.
- The categories were chosen at a level of *abstraction above specific mathematical topics* so they would reflect *process* difficulties.
- For example, some were about students not unpacking a conclusion, as opposed to students having difficulties with quotient groups.

Tentative Categories of Difficulties

We allow categories within categories and hope that their hierarchy will help identify the most important needed interventions. We have thus far identified the following:

- omitting beneficial actions or taking detrimental ones
- inappropriately mimicking a prior proof.
- inadequate proof framework
    - failure to unpack the hypothesis or the conclusion
- unfinished proof
- extraneous statements
- assumption of the negation of a previously established fact

- difficulties with proof by contradiction
- failure to use cases when appropriate
- wrong or improperly used definitions
- insufficient warrant
- incorrect deduction
- assumption of all or part of the conclusion
- assertion of an untrue “result”
- computational errors
- misuse of logic
- nonstandard language/notation.

Examples

- We next give examples – first there will be a sample correct proof.

- This will be followed by an actual student "proof" of the same theorem.

A Sample Correct Proof

Theorem. Let *S* be a semigroup with an identity element *e*. If, for all *s* in *S*, *ss = e*, then *S* is commutative.

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Therefore, S is commutative.

Theorem. Let *S* be a semigroup with an identity element *e*. If, for all *s* in *S*, *ss* = *e*, then *S* is commutative.

Proof:

Let *S* be a semigroup with identity *e*. Suppose for all $s \in S$, $ss = e$.

Let *a*, *b* be elements in S.

Thus $ab = ba$.

Therefore, *S* is commutative.

Theorem. Let S be a semigroup with an identity element e. If, for all s in S, $ss = e$, then S is commutative.

Proof:

Let S be a semigroup with identity e. Suppose for all $s \in S$, $ss = e$.

Let a, b be elements in S.

Now, $abab = e$, so $(abab)b = eb = b$.

Thus $ba = ab$.

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Proof:

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Let a, b be elements in S.

Now, $abab = e$, so $(abab)b = eb = b$.

But $(abab)b = aba(bb) = abae = aba$.

Thus $ba = ab$.

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Let a, b be elements in S.

Now, $abab = e$, so $(abab)b = eb = b$.

But $(abab)b = aba(bb) = abae = aba$.

So, $b = aba$, so $ba = (aba)a = ab(aa) = abe = ab$.

Thus $ba = ab$.

Therefore, S is commutative.

A Student-Constructed "Proof" of the Same Theorem

Student "Proof" 5A.4

Let S be a semigroup with an identity element, e. Let $s \in S$ such that $ss = e$.

Because e is an identity element, $es = se = s$.

Now, $s = se = s(ss)$.

Since S is a semigroup, $(ss)s = es = s$.

Thus $es = se$.

Therefore, S is commutative. QED.

Scratchwork

7.1: A semigroup is called commutative or Abelian if, for each a and $b \in S$, $ab = ba$.

7.5: An element e of a semigroup S is called an identity element of S if,

for all $s \in S$, $es = se = s$.

Student "Proof" 5A.4

Let S be a semigroup with an identity element, e. Let $s \in S$ such that $ss = e$.

Therefore, S is commutative. QED.

The second sentence should have been "Suppose for all $s \in S$, $ss = e$."

With this change, the first part of the framework would have been correct.

Student "Proof" 5A.4

Let S be a semigroup with an identity element, e. Let $s \in S$ such that $ss = e$.

Therefore, S is commutative. QED.

In addition, the student did not finish the framework by introducing arbitrary a and b at the top, followed by "Then $ab = ba$" right above the conclusion.

Student "Proof" 5A.4

Let S be a semigroup with an identity element, e. Let $s \in S$ such that $ss = e$.

Therefore, S is commutative. QED.

Also, the student may not have written the conclusion immediately after writing the hypothesis.

These are beneficial actions *not taken*.

Student "Proof" 5A.4

Let S be a semigroup with an identity element, e. Let $s \in S$ such that $ss = e$.

Therefore, S is commutative. QED.

Had the student written the correct second sentence and taken these two actions, the situation would have been appropriate for exploring and manipulating an object such as *abab*.

Student "Proof" 5A.4

Let S be a semigroup with an identity element, e. Let $s \in S$ such that $ss = e$.

Therefore, S is commutative. QED.

We think that such exploration calls for some self-efficacy, but can lead to a correct proof.

- We consider what the student then wrote, namely,

Because *e* is an identity element, $es = se = s$.

Now, $s = se = s(ss)$.

Since *S* is a semigroup, $(ss)s = es = s$.

Thus $es = se$.

Because e is an identity element, $es = se = s$.

Now, $s = se = s(ss)$.

Since S is a semigroup, $(ss)s = es = s$.

Thus $es = se$.

The first line above violates the mathematical norm of not including definitions that can easily be found outside the proof. Also, it does not move the proof forward.

Because e is an identity element, $es = se = s$.

Now, $s = se = s(ss)$.

Since S is a semigroup, $(ss)s = es = s$.

Thus $es = se$.

The next three lines are not wrong, but do not move the proof forward because to prove commutativity, one needs two arbitrary elements.

These actions are detrimental because they can convince the student that he/she has accomplished something when that is not the case.

Another Sample Correct Proof

Theorem. Let S and T be semigroups and $f{:}S \to T$ be a homomorphism. If G is a subset of S and G is a group with identity e, then $f(G)$ is a group.

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Therefore, $f(G)$ is a group. QED

Theorem. Let S and T be semigroups and $f:S \to T$ be a homomorphism. If G is a subset of S and G is a group with identity e, then $f(G)$ is a group.

Proof:

Let S and T be semigroups and $f:S \to T$ be a homomorphism. Let G be a subset of S and G be a group with identity e.

Part 1: Note that G is a subsemigroup of S so, by Theorem 20.4, $f(G)$ is a semigroup.

Part 2:

Part 3:

Therefore, $f(G)$ is a group. QED

Theorem. Let S and T be semigroups and $f{:}S \to T$ be a homomorphism. If G is a subset of S and G is a group with identity e, then $f(G)$ is a group.

Proof:

Let S and T be semigroups and $f{:}S \to T$ be a homomorphism. Let G be a subset of S and G be a group with identity e.

Part 1: Note that G is a subsemigroup of S so, by Theorem 20.4, $f(G)$ is a semigroup.

Part 2: Let $y \in f(G)$. Then there is $x \in G$ so that $f(x) = y$. Now $f(e) \in f(G)$ and $f(e)y = f(e)\,f(x) = f(ex) = f(x) = y$. Similarly, $y\,f(e) = y$. Thus $f(e)$ is an identity for $f(G)$.

Part 3:

Therefore, $f(G)$ is a group. QED

Theorem. Let S and T be semigroups and $f{:}S\to T$ be a homomorphism. If G is a subset of S and G is a group with identity e, then $f(G)$ is a group.

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Let S and T be semigroups and $f{:}S\to T$ be a homomorphism. Let G be a subset of S and G be a group with identity e.

Part 1: Note that G is a subsemigroup of S so, by Theorem 20.4, $f(G)$ is a semigroup.

Part 2: Let $y \in f(G)$. Then there is $x \in G$ so that $f(x) = y$. Now $f(e) \in f(G)$ and $f(e)y = f(e)\, f(x) = f(ex) = f(x) = y$. Similarly, $y\, f(e) = y$. Thus $f(e)$ is an identity for $f(G)$.

Part 3: Let q in $f(G)$. Then there is $p \in G$ so that $f(p) = q$. Now because G is a group, there is $p' \in G$ so that $pp' = p'p = e$. Thus $q\, f(p') = f(p)\, f(p') = f(pp') = f(e)$, and $f(p')q = f(p')\, f(p) = f(p'\, p) = f(e)$. Thus, each $q \in f(G)$ has an inverse, $f(p')$, in $f(G)$.

Therefore, $f(G)$ is a group. QED.

A Student-Constructed "Proof" of the Same Theorem

Theorem. Let *S* and *T* be semigroups and *f*:*S*→*T* be a homomorphism. If *G* is a subset of *S* and *G* is a group with identity *e*, then *f*(*G*) is a group.

Theorem. Let S and T be semigroups and $f:S \to T$ be a homomorphism. If G is a subset of S and G is a group with identity e, then $f(G)$ is a group.

Student "Proof" 9B.4

Let S and T be semigroups and $f:S \to T$ be a homomorphism.

Suppose $G \subseteq S$ and G is a group with identity e.

Since G is a group and it has identity e, then for each element g in G there is an element g' in G such that $gg' = g'g = e$.

Since f is a homomorphism, then for each element $x \in S$ and $y \in S$, $f(xy)=f(x)f(y)$.

Since $G \subseteq S$, then $f(gg')=f(g)f(g')$. So $f(gg') = f(g'g) = f(e)$. So $f(G)$ has an element $f(e)$ since f is a function.

Therefore, $f(G)$ is a group. QED.

Student "Proof" 9B.4

Let S and T be semigroups and $f{:}S \rightarrow T$ be a homomorphism.

Suppose $G \subseteq S$ and G is a group with identity e.

Therefore, $f(G)$ is a group. QED.

The student has the "first level" framework correct (assuming he/she wrote the last line at the bottom immediately after writing the first two lines).

To complete the framework, the student should have next considered $f(G)$ and noted that there are three things to prove in order for $f(G)$ to be a group. These are actions the student did not take.

Student "Proof" 9B.4

Let S and T be semigroups and $f:S \rightarrow T$ be a homomorphism.

Suppose $G \subseteq S$ and G is a group with identity e.

Since G is a group and it has identity e, then for each element g in G there is an element g' in G such that $gg' = g'g = e$.

Since f is a homomorphism, then for each element $x \in S$ and $y \in S$, $f(xy)=f(x)f(y)$.

Therefore, $f(G)$ is a group. QED.

Instead the student included the definitions of inverse in G and of homomorphism.

These are actions that do not move the proof forward and are detrimental because they can convince the student that something useful has been done.

Student "Proof" 9B.4

Let S and T be semigroups and $f{:}S\rightarrow T$ be a homomorphism.

Suppose $G \subseteq S$ and G is a group with identity e.

Since G is a group and it has identity e, then for each element g in G there is an element g' in G such that $gg' = g'g = e$.

Since f is a homomorphism, then for each element $x \in S$ and $y \in S$, $f(xy)=f(x)f(y)$.

Since $G \subseteq S$, then $f(gg')=f(g)f(g')$. So $f(gg') = f(g'g) = f(e)$. So $f(G)$ has a an element $f(e)$ since f is a function.

Therefore, $f(G)$ is a group. QED.

Perhaps the student was trying to show the existence of an identity and inverses for $f(G)$, but was unsuccessful.

The second student's work may suggest that he/she had some intuitive grasp of the concepts involved. It may be tempting to give partial credit to this student, but from the point of view of having a student learn to construct proofs, doing so may send the "wrong message".

Summarizing, both students

- took a number of actions which they *should not* have taken and
- did *not* take a number of actions which they *should have* taken.

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Thank you
Comments/Questions