A Finite Volume Approach to Multiscale Elasticity

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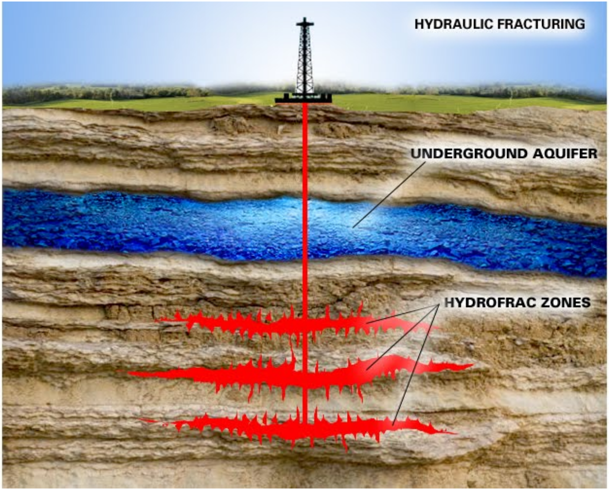
Inspiration

William Kamkwamba, South Africa

Definition

Poroelasticity

Applications



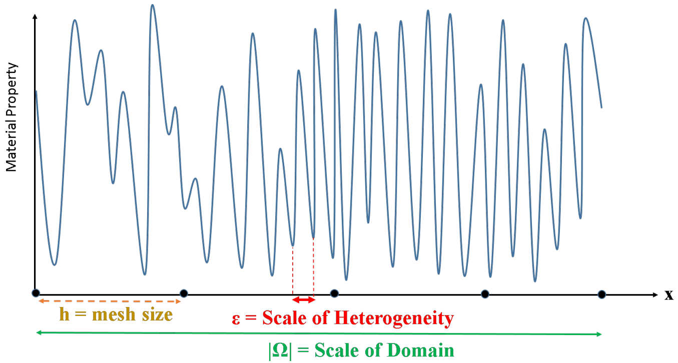
Fluid flow affects solid deformation!

The Challenge

Large variations in material parameters over small spatial scales.

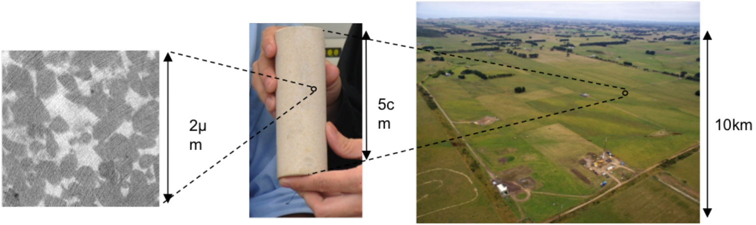
The Goldilocks Problem

Assume $\epsilon << |\Omega|$



- If $h > \epsilon$, then simulation is fast, but highly inaccurate.
- If $h < \epsilon$, then simulation is accurate, but extremely slow.

The Curse of Dimensionality

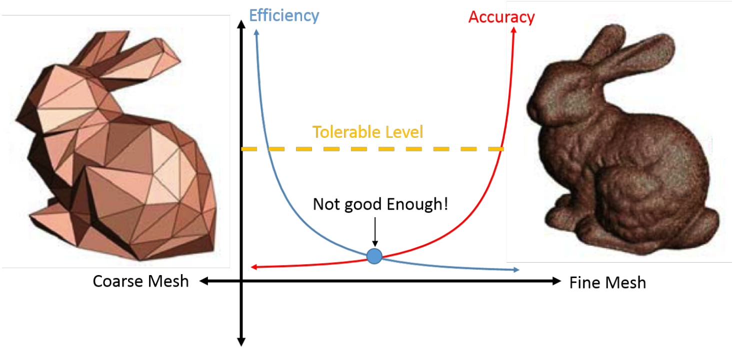


Assuming 10^3 nodes per μm, a Petascale computer solves the equations in

- In 2D $\Rightarrow \approx 3,000$ yrs
- In 3D $\Rightarrow \approx 31$ quadrillion yrs

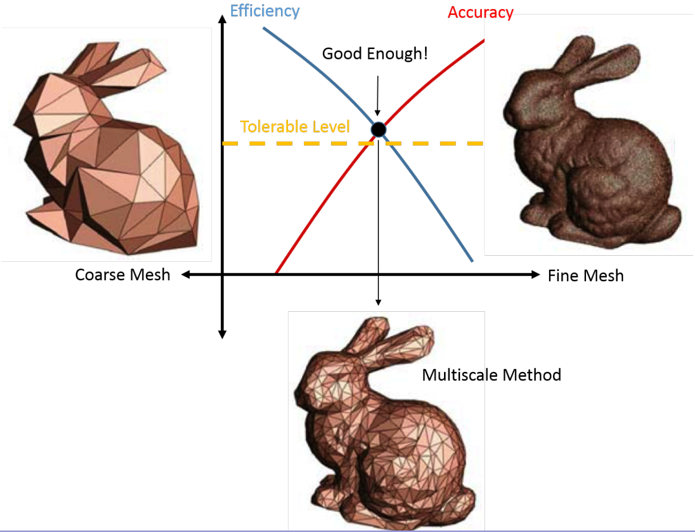
Moral: Parallelization, alone, will not solve this problem!!!

Conventional Methods



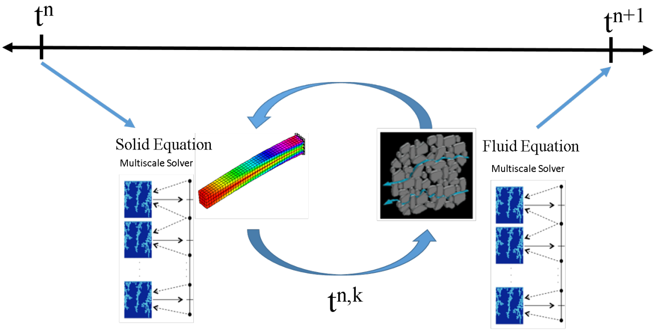
How can we balance the need for accuracy with the need for efficiency?

Multiscale Method



Our approach

Toward a multiscale method for poroelasticity



- Decouple solid & fluid equations
- Develop multiscale methods for each equation

Progress

- Developed & Verified Operator Splitting Method
- Developed 1D Multiscale Flow & Deformation Methods
- Improved methods for neumann conditions & source terms
- **Higher Dimensional method for Fluid Flow**

Today, we demonstrate our multiscale method for the solid equation in higher dimensions

Solid Equation

$$-\nabla \cdot \sigma = \vec{F} \text{ in } \Omega$$

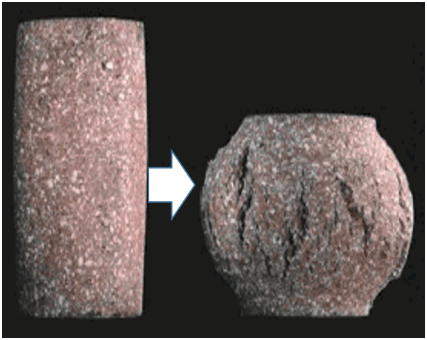
$$\sigma = \sigma(\epsilon)$$

$$\epsilon = \epsilon(\nabla \vec{u})$$

$$\nabla \vec{u} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

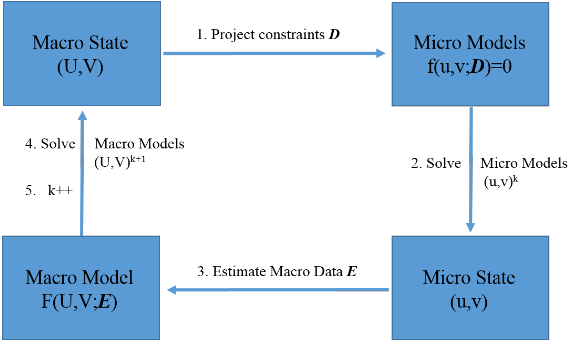
$$u = d(x, y) \text{ on } \partial\Omega_d$$

$$\sigma \cdot n = t(x, y) \text{ on } \partial\Omega_t$$



Momentum balance relates stress to displacement

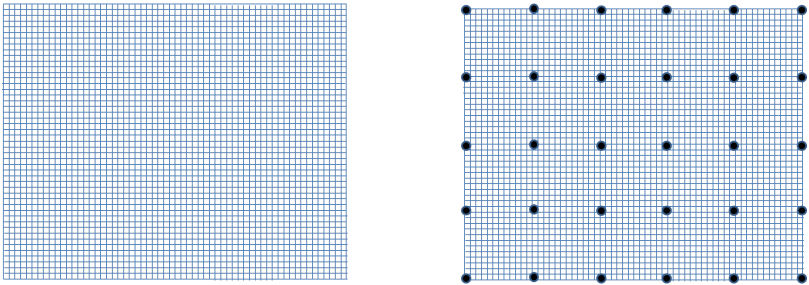
Methodology



Heterogeneous Multiscale Framework (E & Engquist 2003).

Key Idea

Microgrid　　　　Macrogrid



- A fully coupled microscopic model on the entire computational domain Ω
- Seek a solution at a small subset of the microgrid.
- **Key to Efficiency:** Use less info than what is available!

Macro Model

Incomplete Finite Volume Method

$$-\int_{\partial CV} \sigma \cdot \vec{n} = \int_{CV} \vec{F}$$

$$-\int_{CV^E} \sigma_x + \int_{CV^W} \sigma_x - \int_{CV^N} \tau_{xy} + \int_{CV^S} \tau_{xy} = \int_{CV} f \quad (1)$$

$$-\int_{CV^N} \sigma_y + \int_{CV^S} \sigma_y - \int_{CV^E} \tau_{xy} + \int_{CV^W} \tau_{xy} = \int_{CV} g \quad (2)$$

- $\sigma = \begin{bmatrix} \sigma_x & \tau_{x_y} \\ \tau_{xy} & \sigma_y \end{bmatrix}$
- No explicit constitutive relation $\sigma = \sigma(\epsilon(\nabla \vec{u}))$

Micro Model

Linear Heterogeneous Isotropic Model

$$\nabla \cdot \sigma + \vec{F} = 0$$

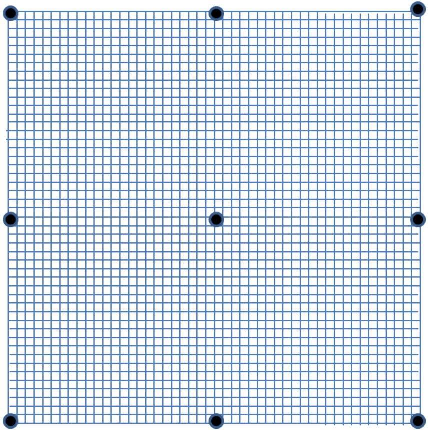
$$\sigma(\epsilon) = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = 2\mu(\vec{x})\epsilon + \lambda(\vec{x}) tr(\epsilon) I$$

$$\epsilon(\nabla \vec{u}) = \frac{1}{2}\left(\nabla \vec{u} + \nabla \vec{u}^T\right)$$

$$\nabla \vec{u} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

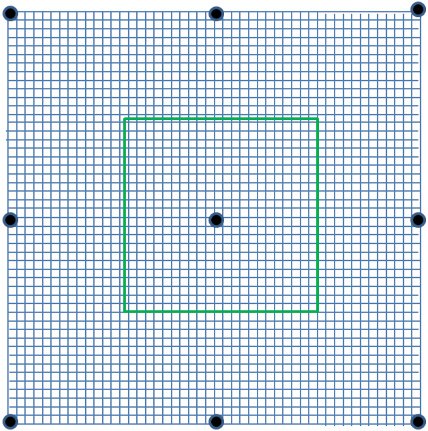
Other models are also possible (molecular dynamics, lattice structures, etc...)

Step 1: Initial Guess



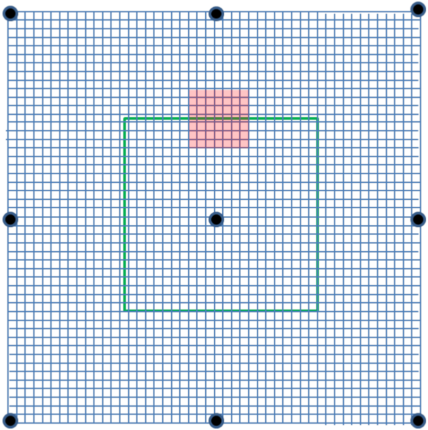
Old field variables $(u_{ij}, v_{ij})^K$

Step 2: Loop



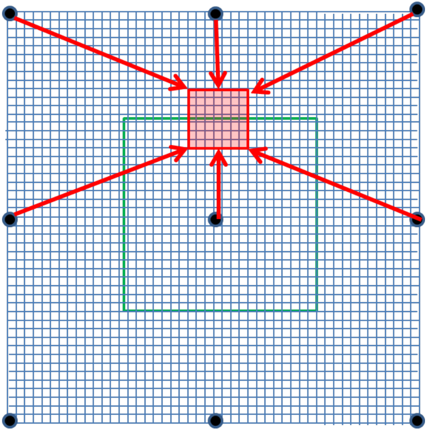
For each control volume boundary *D*

Step 3: Sample



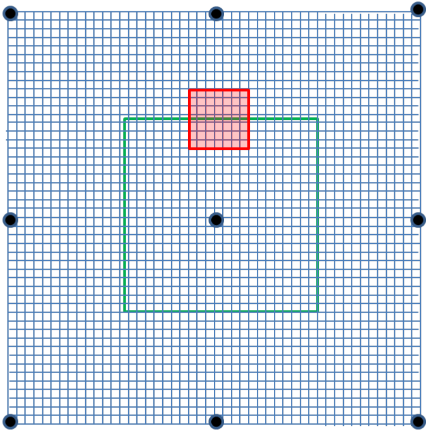
Micro data near CV^D boundary midpoint

Step 4: Constraint Projection



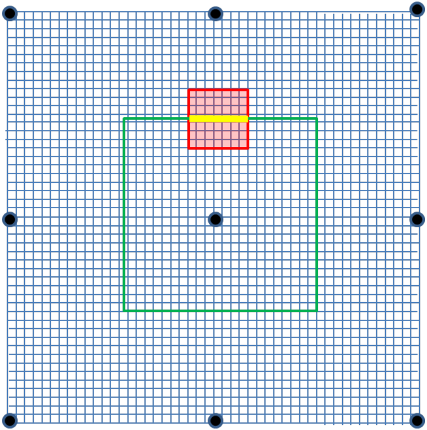
Interpolate BC's from local macro field $(u_{ij}, v_{ij})^K$

Step 5: Solve Micromodel



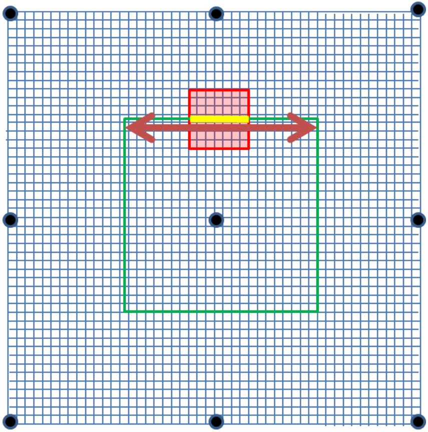
Obtain local micro field in B_δ^D

Step 6: Data Estimation (1)



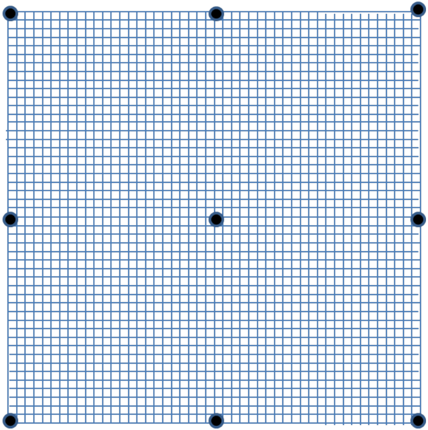
Calculate total normal & shear force along mid cross-section.

Step 6: Data Estimation (2)



Rescale total forces to entire control volume boundary *D*

Step 7: Solve Macro Model



Obtain updated field variables $(u_{ij}, v_{ij})^{K+1}$

Key to Micro-Macro Iterations

- Assume $\sigma = 0$ when $\nabla \vec{u} = 0$.
- Assume u_x, u_y, v_x and v_y are *independent variables*.
- Taylor series expansion of σ_y, σ_x, and τ_{xy}
- Fixed Point Iteration over K

Then

$$\int_{\partial CV^D} \nu^{K+1} = \sum_{i=1}^{4} \frac{\int_{\partial CV^D} \nu^{D,K}\left(G_i^{D,K}\right)}{G_i^{D,K} \cdot e_i} G_i^{D,K+1} \cdot e_i \tag{3}$$

- **Stress Component:** $\nu = \sigma_y, \sigma_x$, and τ_{xy}
- **Boundary:** $D = N, S, E, W$
- **Subgradient:**[1] $G_i^{D,K} \equiv vec\left(\nabla \vec{u}^{D,K}\right) \circ e_i$ (i=1,...,4)

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[1] $(\circ)$denotes the **Hadamard Product** and e_i denotes **standard basis** in $\mathbb{R}^4$

Numerical Experiments

Unit Square Domain $\Omega = [0, 1]^2$
Cases w/ Analytical Solutions

- Prescribed displacement u, v functions
- Smooth material functions λ, μ
- Derived source terms f, g

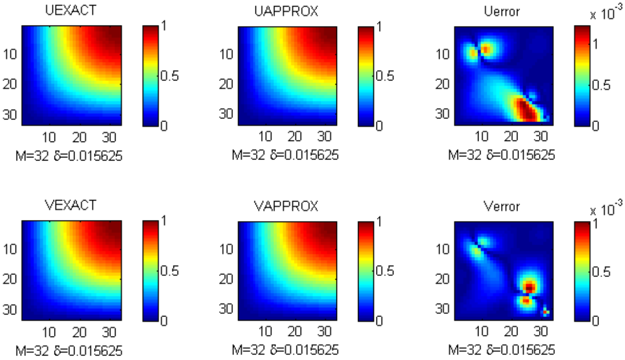
Cases w/o Analytical Solutions

- Random material parameters λ, μ
- Prescribed source terms f, g
- Reference Solution obtained numerically

Analyze convergence as the total sampling area $\to |\Omega|$

Results - Analytical Case

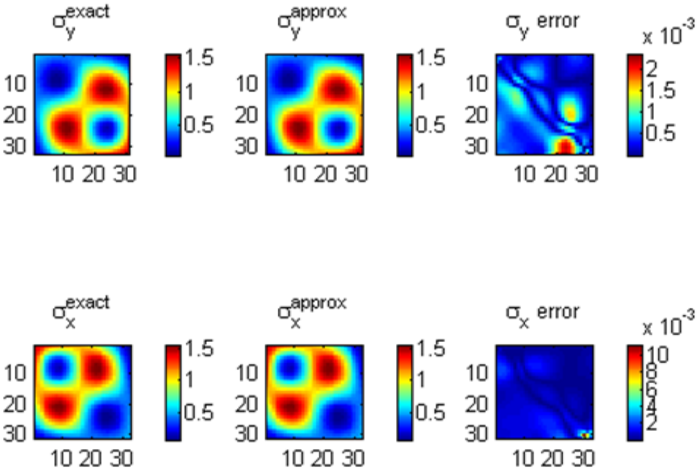
Displacement



$$u = v = sin(\frac{\pi x}{2})sin(\frac{\pi y}{2}),\ \lambda = \mu = 11 + sin(2\pi x)sin(2\pi y)$$

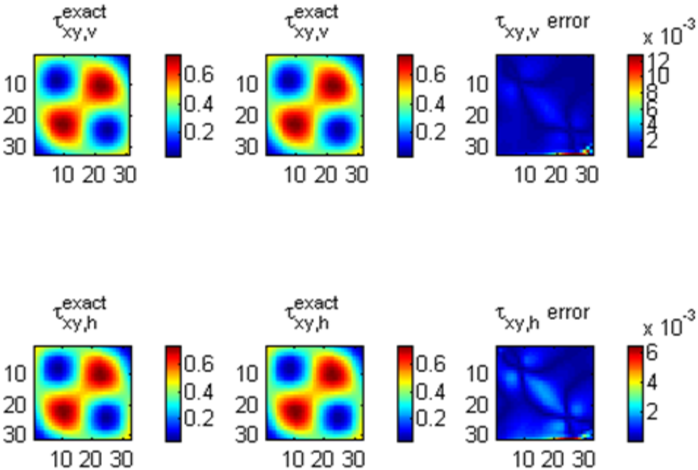
Results - Analytical Case

Normal Stress



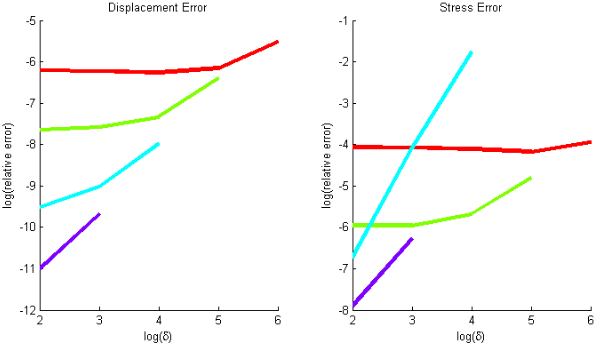
Results - Analytical Case

Shear Stress



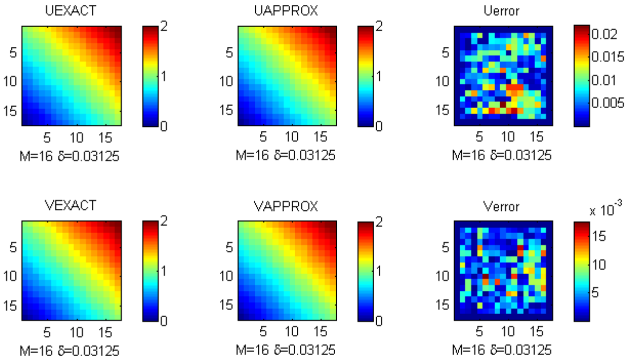
Results - Analytical Case

Convergence



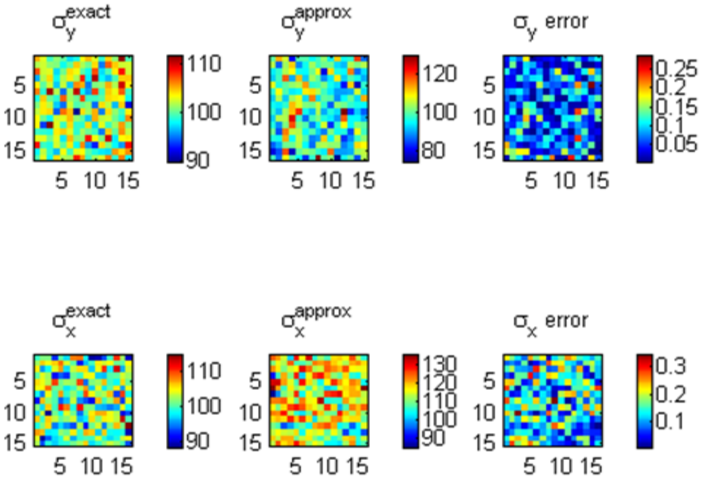
Results - Random Case

Displacement



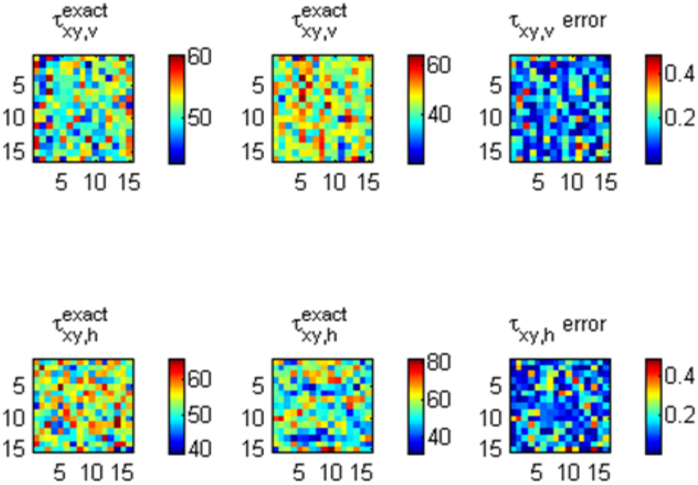
Results - Random Case

Normal Stress



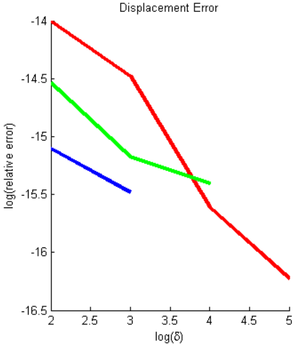
Results - Random Case

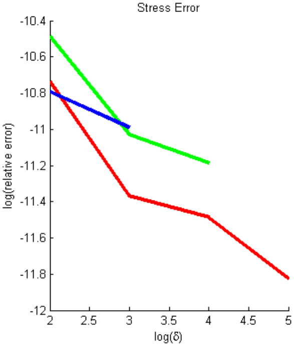
Shear Stress



Results - Random Case

Convergence





Conclusions

- Our method fails as a general purpose PDE solver
- Works best in the worse case scenario: ***random heterogeneity***
- Displacement is well approximated, but not stress.
- Algorithm is highly **parallelizable**
- Results are consistent with other implementations of HMM.

Future Work

- Multiphysics Simulation
- Parallelization
- Improve stress estimation
- Test with other micromodels