Existence and asymptotic behaviour of solutions to second-order evolution equations of monotone type

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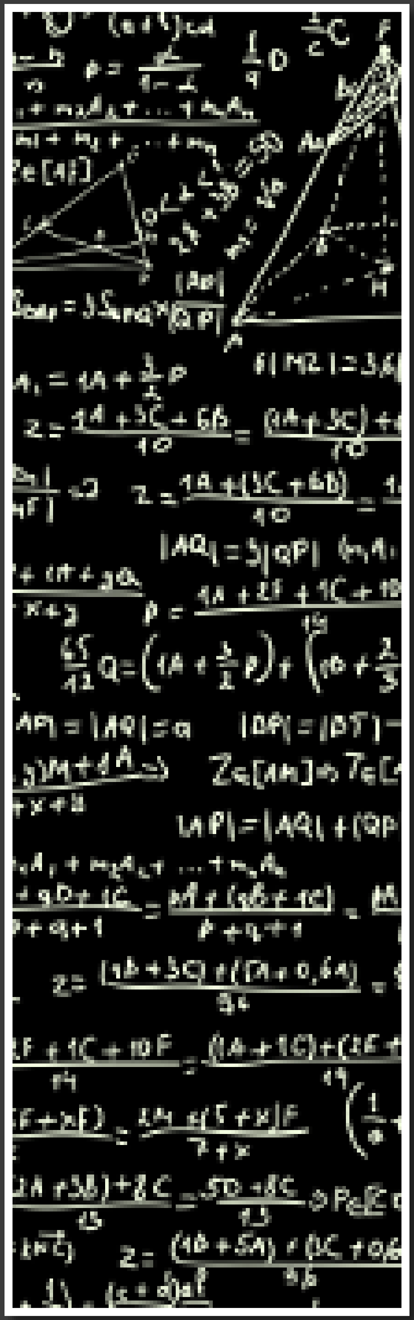
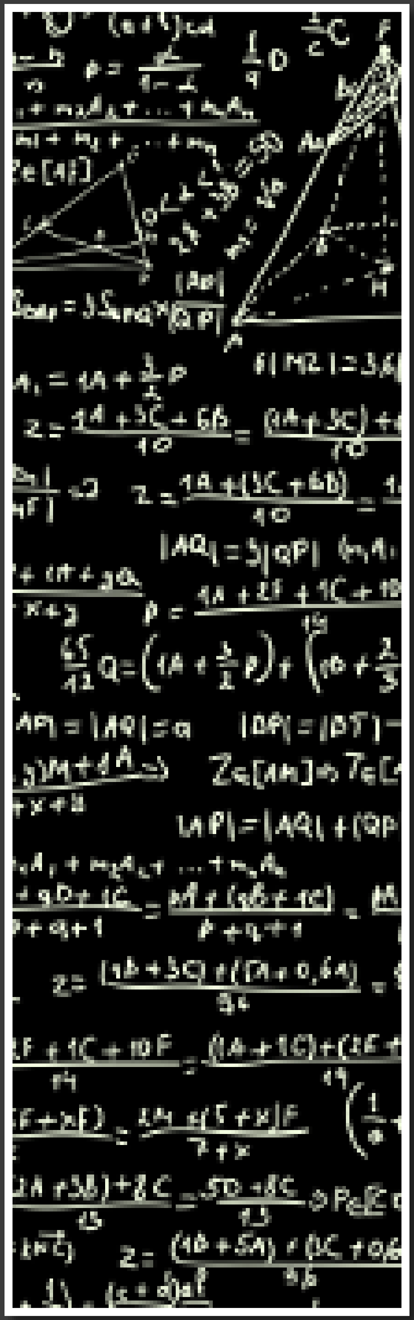
15th Joint UTEP/NMSU Workshop on Mathematics, Computer Science, and Computational Sciences

Klara Loos



AGENDA

1. The second order differential equation
2. Recent results on existence of solutions
3. Recent results on asymptotic behaviour of solutions
4. Discussion



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1. The second order differential equation
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The second-order differential equation

second-order

positive half-line

$$p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t) \quad for\ a.a. \quad t \in \mathbb{R}_+ := [0, \infty) \quad \text{(E)}$$

incomplete evolution problem

The second-order differential equation

second-order

positive half-line

$$p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t) \quad for\ a.a. \quad t \in \mathbb{R}_+ := [0, \infty) \qquad \text{(E)}$$

with the condition

$$u(0) = x \in \overline{D(A)} \qquad \text{(B)}$$

initial data

(H1) conditions for A

(H2) conditions for p, q

- Monotone type: A maximal monotone operator
- Homogenous: f(t) = 0
- p,q are constants
- p,q are real functions, …
- …

The second-order differential equation

second-order

positive half-line

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initial data

(H1) conditions for A

(H2) conditions for p, q

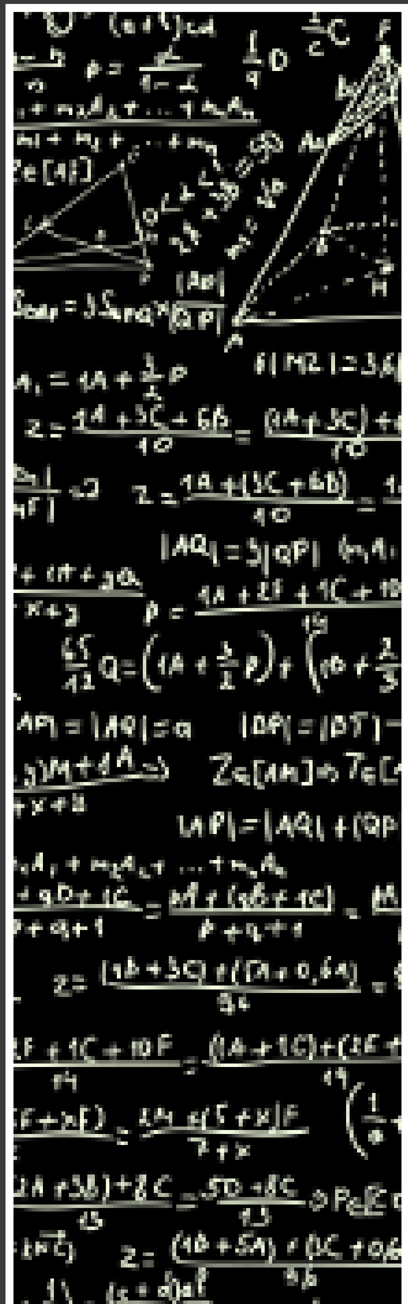
A in Hilbert space H is monotone:

$$\langle Ax_1 - Ax_2, x_1 - x_2 \rangle \geq 0 \qquad \forall x_1, x_2 \in D(A)$$

A is maximal monotone:

A is maximal monotone, if it is maximal in the set of monotone operators. $\Leftrightarrow R(I + \lambda A) = H, \quad \forall \lambda > 0$

[Brézis, 2010]



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1. The second order differential equation
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Recent Results: Existence and uniqueness of a bounded solution

$$p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t) \quad for\ a.a. \quad t \in \mathbb{R}_+ := [0, \infty) \qquad \text{(E)}$$

with the condition

$$u(0) = x \in \overline{D(A)} \qquad \text{(B)}$$

Where A is a maximal monotone operator in a real Hilbert space H.

p,q are real valued functions defined on [0,∞).

(H1) $A: D(A) \subset H \to H,$ $\quad A$ maximal monotone operator

(H2) $p, q \in L^{\infty}(\mathbb{R}_+),$ $\quad ess \inf p > 0,$ $\quad q^+ \in L^1(\mathbb{R}_+), q^+ = \max\{q(t), 0\}$

[E1] G. Moroşanu, *Existence results for second-order monotone differential inclusions on the positive half-line*, J.Math. Appl. 419 (2014) 94-113

Recent Results: Existence and uniqueness of a bounded solution

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Existence and uniqueness of bounded solution

- non-homogenous $f(t) \neq 0$
- Non-constant functions q,p & mild condition

$p(t) \geq \alpha > 0$

$L^{\infty}(\mathbb{R}_+) :=$ space, of essential bounded functions

[E1] G. Moroşanu, *Existence results for second-order monotone differential inclusions on the positive half-line*, J.Math. Appl. 419 (2014) 94-113

Existence of a unique bounded solution
Case: $p \equiv 1,\ q \equiv 0,\ f \equiv 0$

$$u''(t) \in Au(t) \quad for\ a.a. \quad t \in \mathbb{R}_+ := [0, \infty) \qquad \text{(E)}$$

with the condition

$$u(0) = x \in \overline{D(A)} \qquad \text{(B)}$$

Where A is a maximal monotone operator in a real Hilbert space H.

(H1) $A: D(A) \subset H \rightarrow H,$ A maximal monotone operator

[E2] V. Barbu, *Sur un problème aux limites pour une classe d'équations differentielles nonlinéaires abstraites du deuxièmes ordre en t,* C.R. Accad. Sci. Paris 27 (1972) 459 - 462
[E3] V. Barbu, *A clase of boundary problems for second-order abstract differential equations,* J. Fae. Sci. Univ. Tokyo, Sect. I 19 (1972) 295-319

Existence of a unique bounded solution Homogenous case: $f \equiv 0$

$$p\,u''(t) + q\,u'(t) \in Au(t) \quad for\ a.a. \quad t \in \mathbb{R}_+ := [0, \infty) \qquad \text{(E)}$$

with the condition

$$u(0) = x \in \overline{D(A)} \qquad \text{(B)}$$

Where A is a m-accretive operator in a real Banach space.

(H1) $p\,, q \in \mathbb{R}_+$ are constants

[E4] H.Brezis, *Équations d'évolution du second ordre associées à des opérateurs monotones*, Isreal J. Math. 12 (1972) 51-60.
[E5] N. Pavel, *Boundary value problems on* $[0, +\infty]$ *for second-order differential equations associated to monotone operators in Hilber spaces,* in: Proceedings of the Institute of Mathematics Iasi (1974), Editura Acad. R. S. R., Bucharest, 1976, pp.145-154.
[E6] L. Véron, *Problèmes d'évolution du second ordre associées à des opérateurs monotones, C.R. Acad. Sci. Paris 278* (1974) 1099-1101.
[E7] L. Véron, *Equations d'evolution du second ordre associées à des opérateurs maximaux monotones*, Proc. Roy. Soc. Edinburgh Sect. A 75 (2) (1975/1976) 131-147.
[E8] E.I. Poffald, S.Reich, *An incomplete Cauchy problem*, J. Math. Anal. Appl. 113 (2) (1986) 514-543.

Recent Results: Existence and uniqueness of a bounded solution

$$p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t) \quad for\ a.a. \quad t \in \mathbb{R}_+ := [0, \infty) \qquad \text{(E)}$$

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p,q are real valued functions defined on [0,∞).

(H1) $A: D(A) \subset H \to H, \quad A$ maximal monotone operator

(H2) $p, q \in L^\infty(\mathbb{R}_+), \ \ ess \inf p > 0, q^+ \in L^1(\mathbb{R}_+), q^+ = \max\{q(t), 0\}$

Existence and uniqueness of bounded solution

- non-homogenous f(t) ≠ 0
- Non-constant functions q,p & mild condition

[1] G. Moroşanu, *Existence results for second-order monotone differential inclusions on the positive half-line*, J.Math. Appl. 419 (2014) 94-113

Development: Existence and uniqueness of a bounded solution

- $p \equiv 1, q \equiv 0,$
- nonhomogeneous $f \equiv 0$
- A is a maximal monotone operator in a real Hilbert space H.

- $p, q \in \mathbb{R}_+$ are constants
- homogenous case: $f \equiv 0$
- $A = m$-accretive operator in a real Banach space.

- $p(t), q(t) \in L^\infty(\mathbb{R}_+)$
- non-homogenous f(t) $\neq 0$
- non-constant functions q,p & mild condition

[E1, E2] [E4, E5, E6, E7, E8] [1]

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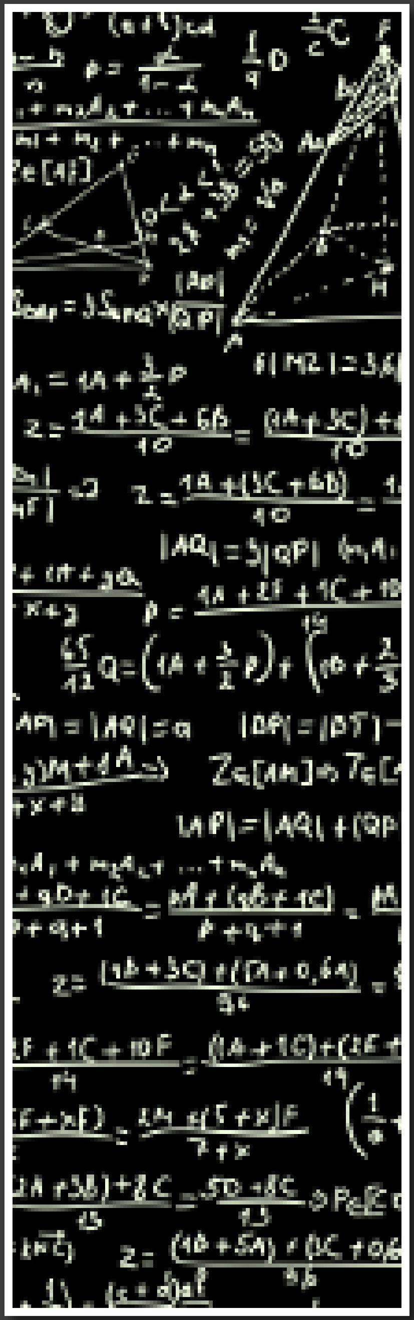

AGENDA

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2. Recent results on existence of solutions
3. Recent results on asymptotic behaviour of solutions
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RECENT RESULTS: Asymptotic behaviour

Nonlinear second order evolution equation:

$$p(t)u''(t) + r(t)u'(t) \in Au(t) \quad for\ a.a. \quad t \in \mathbb{R}_+ := [0, \infty) \qquad \text{(E)}$$

with the condition

$$u(0) = u_0, \qquad \sup|\mathrm{u(t)}| < +\infty \qquad \text{(B)}$$

Where A is a maximal monotone operator in a real Hilbert space H.

[3] B. Djafari-Rouhani, H. Khatibzadeh, *A note on the strong convergence of solutions to a second order evolution equation*, J. Math. Anal. Appl. 401 (2013) 963–966.

RECENT RESULTS: Asymptotic behaviour

$f \equiv 0$ | p, r time dependant

Nonlinear second order evolution equation:

$$p(t)u''(t) + r(t)u'(t) \in Au(t) \quad for\ a.a. \quad t \in \mathbb{R}_+ := [0, \infty) \qquad \text{(E)}$$

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bounded solution

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with the condition

$$u(0) = u_0, \qquad \sup|u(t)| < +\infty \qquad \text{(B)}$$

Where A is a maximal monotone operator in a real Hilbert space H.

- Strong convergence for $t \to \infty$: $u(t) \to p \in A^{-1}(0)$

$0 \in A(p) \subset H$

- rate of convergence: $|u(t) - p| = O(\int_t^\infty e^{-\int_0^s \frac{r(\tau)}{2p(\tau)} d\tau} ds)$

- $u(t) \to p \in A^{-1}(0) \Rightarrow A^{-1}(0) \neq \emptyset$

  (not assumed, now a consequence)

[3] B. Djafari-Rouhani, H. Khatibzadeh, *A note on the strong convergence of solutions to a second order evolution equation*, J. Math. Anal. Appl. 401 (2013) 963–966.

RECENT RESULTS: Asymptotic behaviour

Theorem 2.1. [3]

u(t) is solution of (E), (B)

- ✓ $\int_0^\infty e^{-\int_0^s \frac{r(\tau)}{2p(\tau)} d\tau} ds < \infty$
- ✓ $r'(t) \leq 0$

conditions

⇒

I. $u(t) \to p \in A^{-1}(0) \neq \emptyset$

II. $|u(t) - p| = O(\int_t^\infty e^{-\int_0^s \frac{r(\tau)}{2p(\tau)} d\tau} ds)$

consequences

[3] B. Djafari-Rouhani, H. Khatibzadeh, *A note on the strong convergence of solutions to a second order evolution equation*, J. Math. Anal. Appl. 401 (2013) 963–966.

Example

Ordinary differential equation

$$p(t) = 1, r(t) = \frac{3}{t+1}$$

$$u'' + \frac{3}{t+1}u' = 3u^{\frac{5}{3}}, \qquad t \in \mathbb{R}_+$$

$$Au = 3u^{\frac{5}{3}}$$

Theorem 2.1, [3]

u(t) is solution of (E), (B)

conditions:

$$\int_0^\infty e^{-\int_0^s \frac{r(\tau)}{2p(\tau)}d\tau} ds < \infty$$

$$r'(t) \leq 0$$

consequences:

$\mathrm{u(t)} \to p \in A^{-1}(0) \neq \emptyset$

$$|u(t) - p| = O\left(\int_t^\infty e^{-\int_0^s \frac{r(\tau)}{2p(\tau)}d\tau} ds\right)$$

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conditions

$$u(t) \to p \in A^{-1}(0) \neq \emptyset$$

$$|u(t) - p| = O\left(\int_t^\infty e^{-\int_0^s \frac{r(\tau)}{2p(\tau)}d\tau} ds\right)$$

consequences

Verify solution $u(t) = \frac{1}{(t+1)^3}$:

$$u'(t) = \frac{-3}{(t+1)^4}$$

$$u''(t) = \frac{12}{(t+1)^5}$$

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$$|u(t) - p| = O\left(\int_t^\infty e^{-\int_0^s \frac{r(\tau)}{2p(\tau)}d\tau} ds\right)$$

consequences

$$p(t) = 1, r(t) = \frac{3}{t+1}$$

$$u'' + \frac{3}{t+1}u' = 3u^{\frac{5}{3}}, \qquad t \in \mathbb{R}_+$$

Verify solution $u(t) = \frac{1}{(t+1)^3}$:

$$u'(t) = \frac{-3}{(t+1)^4}$$

$$u''(t) = \frac{12}{(t+1)^5}$$

$$u'' + \frac{3}{t+1}u' = \frac{12}{(t+1)^5} - \frac{3}{(t+1)}\frac{3}{(t+1)^4} = \frac{3}{(t+1)^5} = 3u^{\frac{5}{3}} \checkmark$$

Example

Ordinary differential equation

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Conditions:

$$\int_0^\infty e^{-\int_0^s \frac{r(\tau)}{2p(\tau)} d\tau} ds < \infty$$

$$r'(t) \le 0$$

Consequences:

$$u(t) \to p \in A^{-1}(0) \neq \emptyset$$

$$|u(t) - p| = O\left(\int_t^\infty e^{-\int_0^s \frac{r(\tau)}{2p(\tau)} d\tau} ds\right)$$

Verify condition I:

$$r'(t) = -\frac{3}{(t+1)^2} \le 0$$ ✓

Example

Ordinary differential equation

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$$u'' + \frac{3}{t+1}u' = 3u^{\frac{5}{3}}, \qquad t \in \mathbb{R}_+$$

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$$\int_0^\infty e^{-\int_0^s \frac{r(\tau)}{2p(\tau)}d\tau} ds < \infty$$

$$r'(t) \leq 0$$

conditions

$$\mathrm{u(t)} \to p \in A^{-1}(0) \neq \emptyset$$

$$|u(t) - p| = O\left(\int_t^\infty e^{-\int_0^s \frac{r(\tau)}{2p(\tau)}d\tau} ds\right)$$

consequences

Verify condition II:

$$\int_0^\infty e^{-\int_0^s \frac{3}{2(\tau+1)}d\tau} ds = \int_0^\infty e^{-\left[\frac{3}{2}\ln(\tau+1)\right]_0^s} ds = \int_0^\infty e^{-\frac{3}{2}\ln(s+1)} ds$$

Example

Ordinary differential equation

$$p(t) = 1, r(t) = \frac{3}{t+1}$$

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consequences

Verify condition II:

$$\int_0^\infty e^{-\int_0^s \frac{3}{2(\tau+1)}d\tau} ds = \int_0^\infty e^{-\left[\frac{3}{2}\ln(\tau+1)\right]_0^s} ds = \int_0^\infty e^{-\frac{3}{2}\ln(s+1)} ds$$

$$= \int_0^\infty (s+1)^{-\frac{3}{2}} ds = \lim_{\beta\to\infty}\left[-\frac{2}{(s+1)^{\frac{1}{2}}}\right]_0^\beta = 0 + 2 < \infty \quad ✓$$

Example

Ordinary differential equation

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u(t) is solution of (E), (B)

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$$r'(t) \le 0$$

conditions

$$u(t) \to p \in A^{-1}(0) \neq \emptyset$$

$$|u(t) - p| = O\left(\int_t^\infty e^{-\int_0^s \frac{r(\tau)}{2p(\tau)}d\tau} ds\right)$$

consequences

$$r'(t) = -\frac{3}{(t+1)^2} \le 0 \quad ✓$$

$$\int_0^\infty e^{-\int_0^s \frac{3}{2(\tau+1)}d\tau} ds = 2 < \infty \quad ✓$$

conditions

Example

Ordinary differential equation

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conditions

$$\text{u(t)} \to p \in A^{-1}(0) \neq \emptyset$$

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consequences

$$r'(t) = -\frac{3}{(t+1)^2} \leq 0 \quad ✓$$

$$\int_0^\infty e^{-\int_0^s \frac{3}{2(\tau+1)}d\tau} ds = 2 < \infty \quad ✓$$

conditions

$$\text{u(t)} = \frac{1}{(t+1)^3} \xrightarrow[t\to\infty]{} 0 \quad ✓$$

$$0 = A(0) \neq \emptyset$$

$$|u(t) - p| = O\left(\int_t^\infty e^{-\int_0^s \frac{3}{2(\tau+1)}d\tau} ds\right)$$

consequences

OUTLOOK: Further research & development

Existence of solutions

Time asymptotics

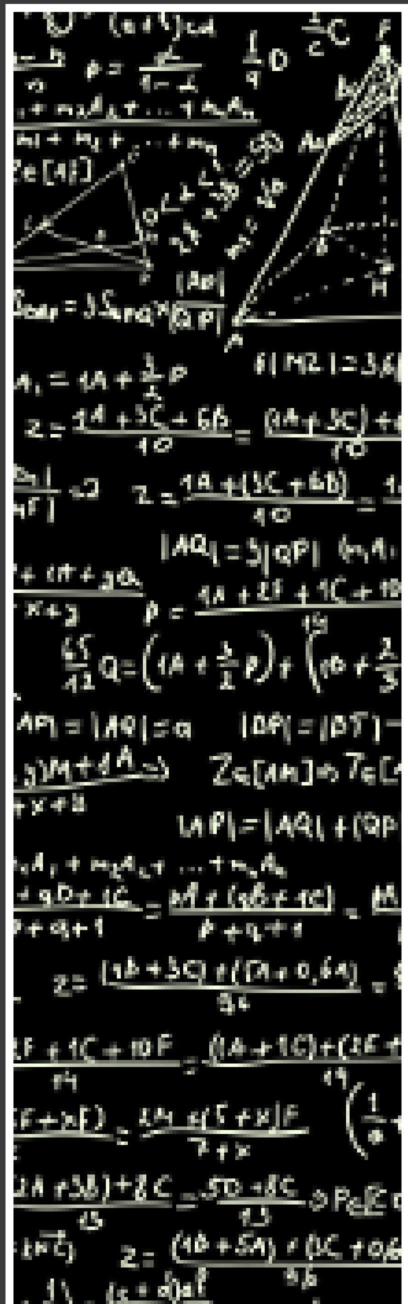
OUTLOOK:
Further research & development

Existence of solutions

Time asymptotics

- More generality
- Higher convergence
- Milder conditions on p(t), r(t)
- Less assumptions

2014



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(1) G. Moroşanu, *Existence results for second-order monotone differential inclusions on the positive half-line*, J. Math. Anal. Appl. 419 (2014) 94-113.

(2) B. Djafari-Rouhani, H. Khatibzadeh, *A strong convergence theorem for solutions to a nonhomogeneous second order evolution equation*, J. Math. Anal. Appl. 363 (2010) 648–654.

(3) B. Djafari-Rouhani, H. Khatibzadeh, *A note on the strong convergence of solutions to a second order evolution equation*, J. Math. Anal. Appl. 401 (2013) 963–966.

(4) B. Djafari-Rouhani, H. Khatibzadeh, *Asymptotic Behavior for a General Class of Homogeneous Second Order Evolution Equations in a Hilbert Space*, To appear in Dynamic Systems and Applications

THANK YOU FOR YOUR ATTENTION

Please give a comment in order to discuss about the topic!

BACKUP:

Development: Asymptotic behaviour

Development – Overview timeline

$p(t)u''(t) + r(t)u'(t) \in Au(t)$
A maximal monotone
$p \in W^{2,\infty}(0,+\infty), r \in W^{1,\infty}(0,+\infty)$

$p \equiv 1, r \equiv 0$: $u''(t) \in Au(t)$
Hilbert space

$p \equiv 1, r \equiv 0$: $u''(t) \in Au(t)$
Banach space

Strong convergence
$p(t)u''(t) + r(t)u'(t) \in Au(t)$
$u(0) = u_0, \quad \sup |u(t)| < +\infty$
+ Convergence rate

[A1, A2] [A3] [A4, A5]

Existence and uniqueness of a bounded solution

74/75 75/76 76 80 82 85/86 85 88 98 2007 09 10 2013 [3]

Asymptotic behaviour

[A6] [A7] [A6,A8, A9, A10, A11, A12] [A13, A14, A15] [A17]

weak convergence
$p = 1, r = 0$: $u''(t) \in Au(t)$

strong convergence
does not hold in general

Strong convergence
with additional
assumptions on the
maximal monotone
operator A

$p(t) = 1, f(t) \neq 0$
$r(t) = c \le 0$
weak and strong
convergence
theorems

$A = \partial\varphi$
+ weak and strong
convergence theorems

Development: Asymptotic behavior

1. Véron proved the existence of solutions to the second order evolution equation:

$$p(t)u''(t) + r(t)u'(t) \in Au(t) \quad for\ a.a. \quad t \in \mathbb{R}_+ := [0, \infty) \qquad \text{(E)}$$

with the condition

$$u(0) = u_0, \qquad \sup|u(t)| < +\infty, \qquad t \geq 0 \qquad \text{(B)}$$

where A is a maximal monotone operator, following conditions hold

(C1) $p \in W^{2,\infty}(0, +\infty), \quad r \in W^{1,\infty}(0, +\infty)$

(C2) $\exists\ \alpha > 0$, such that $\forall t \geq 0,\ p(t) \geq \alpha$

Additional results from Véron for solution u:

- ✓ $u' \in H^1\big((0, \infty); H\big)$
- ✓ u is unique if $\int_0^\infty e^{-\int_0^t \frac{r(s)}{p(s)} ds} dt = +\infty$

[A1] L. Véron, *Problèmes d'evolution du second ordre associés à opérateurs monotones*, C. R. Acad. Sci. Paris Sér. A 278 (1974) 1099-1101.

[A2] L. Véron, *Equations d'évolution du second ordre associés à opérateurs maximaux monotones*, Proc. Roy. Soc. Edinburgh Sect. A 75 (1975-1976) 131-147.

Development: Asymptotic behavior

homogenous

1. Véron proved the existence of solutions to the second order evolution equation:

$$p(t)u''(t) + r(t)u'(t) \in Au(t) \quad for\ a.a. \quad t \in \mathbb{R}_+ := [0, \infty) \qquad \text{(E)}$$

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where A is a maximal monotone operator, following conditions hold

(C1) $p \in W^{2,\infty}(0,+\infty), \quad r \in W^{1,\infty}(0,+\infty)$

Sobolev spaces

(C2) $\exists\ \alpha > 0$, such that $\forall t \geq 0,\ p(t) \geq \alpha$

$p(t) \geq \alpha > 0$

Additional results from Véron for solution u:

- ✓ $u' \in H^1\big((0,\infty); H\big)$

$u', u'' \in L^2$

Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$

- ✓ u is unique if $\int_0^\infty e^{-\int_0^t \frac{r(s)}{p(s)}ds} dt = +\infty$

[A1] L. Véron, *Problèmes d'evolution du second ordre associés à opérateurs monotones*, C. R. Acad. Sci. Paris Sér. A 278 (1974) 1099-1101.

[A2] L. Véron, *Equations d'évolution du second ordre associées à opérateurs maximaux monotones*, Proc. Roy. Soc. Edinburgh Sect. A 75 (1975-1976) 131-147.

Development: Asymptotic behavior

2. Barbu proved the existence of solutions to the second order evolution equation:
$p \equiv 1, r \equiv 0$

$$u''(t) \in Au(t) \quad for\ a.a. \quad t \in \mathbb{R}_+ := [0, \infty) \qquad \text{(E)}$$

with the condition

$$u(0) = u_0, \qquad \sup |u(t)| < +\infty, \qquad t \geq 0 \qquad \text{(B)}$$

Where A is a maximal monotone operator in Hilbert spaces.

[A3] V.Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishing, Leiden, 1976

Development: Asymptotic behavior

3. Poffald and Reich proved the existence of solutions to the second order evolution equation:

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[A4] E.I. Poffald, S. Reich, An incomplete Cauchy problem, J. Math. Anal. Appl. 113 (1986) 514-543

[A5] E.I. Poffald, S. Reich, A quasi-autonomous second-order differential inclusion, in: Trends in the Theory and Practice of Nonlinear Analysis (Arlington, Tex., 1984), in: North-Holland Math. Stud., vol. 110, North-Holland, Amsterdam, 1985, pp. 387-392.

Development: : Asymptotic behavior

3. Poffald and Reich proved the existence of solutions to the second order evolution equation:

$p = 1, r = 0$

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Where A is a maximal monotone operator in Banach spaces.

4. Bruck proved weak convergence in this special case.

[A6] R.E. Bruck, Asymptotic convergence of nonlinear contraction semigroups in Hilbert spaces, J. Funct. Anal. 18 (1975) 15-26

Development: : Asymptotic behavior

3. Poffald and Reich proved the existence of solutions to the second order evolution equation:
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4. Bruck proved weak convergence in this special case.

5. Véron showed that the strong convergence does not hold in general for a maximal monotone operator A, for $p(t) \equiv 1, r(t) \equiv 0$.

[A7] L. Véron, Un exemple concernant le comportement asymptotique de la solution bornée de l'équation $\frac{d^2}{dt^2} \in \partial\varphi(u)$, Monatsh. Math. 89 (1980) 57 -67.

Development: : Asymptotic behavior

6. Bruck, Mitidieri, Morosanu, Apreutesei: Strong convergence of solutions with additional assumptions on the maximal monotone operator A

[A6] R.E. Bruck, Asymptotic convergence of nonlinear contraction semigroups in Hilbert spaces, J. Funct. Anal. 18 (1975) 15-26
[A8] E. Mitidieri, Asymptotic behaviour of some second order evolution equations, Nonlinear Anal. 6 (1982) 1245–1252.
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Development: Asymptotic behavior

6. Bruck, Mitidieri, Morosanu, Apreutesei: Strong convergence of solutions with additional assumptions on the maximal monotone operator A

7. Djafari-Rouhani, Khatibzadeh: Strong convergence of solutions to (E), (B) when $p(t) \equiv 1, r(t) \equiv c \leq 0$

\+ extend previous results to nonhomogeneous case without assumptions $A^{-1}(0) \neq \emptyset$ or A as maximal monotone operator

[A13] B. Djafari Rouhani, H. Khatibzadeh, Asymptotic behavior of bounded solutions to a class of second order nonhomogeneous evolution equations, Nonlinear Anal. 70 (2009) 4369–4376.

[A14] B. Djafari Rouhani, H. Khatibzadeh, Asymptotic behavior of bounded solutions to a nonhomogeneous second order evolution equation of monotone type, Nonlinear Anal. 71 (2009) e147–e152.

[A15] B. Djafari Rouhani, H. Khatibzadeh, Asymptotic behavior of bounded solutions to some second order evolution systems, Rocky Mountain J. Math. 40 (2010) 1289–1311.

[A16] B. Djafari Rouhani, H. Khatibzadeh, *A strong convergence theorem for solutions to a nonhomogenous second order evolution equation*, J. Math. Anal. Appl. 363 (2010) 648-654.

Development: Asymptotic behavior

6. Bruck, Mitidieri, Morosanu, Apreutesei: Strong convergence of solutions with additional assumptions on the maximal monotone operator A

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+ extend previous results to nonhomogeneous case without assumptions $A^{-1}(0) \neq \emptyset$ or A as maximal monotone operator

8. Djafari-Rouhani, Khatibzadeh: $A = \partial\varphi$ is the subdifferential of a proper, convex and lower semicontinuous function φ

+ proof of an ergodic theorem

+ weak and strong convergence theorems to (E),(B) with additional assumptions on *r(t)* and A

[A17] B. Djafari Rouhani, H. Khatibzadeh, Asymptotic behavior of solutions to some homogeneous second order evolution equations of monotone type, J. Inequal. Appl. (2007) 8. Art. ID 72931.

RECENT RESULTS: Asymptotic behavior

Strong convergence of solutions to the nonlinear second order evolution equation:

$$p(t)u''(t) + r(t)u'(t) \in Au(t) \quad for\ a.a. \quad t \in \mathbb{R}_+ := [0, \infty) \qquad \text{(E)}$$

with the condition

$$u(0) = u_0, \qquad \sup|\mathrm{u(t)}| < +\infty \qquad \text{(B)}$$

Where A is a maximal monotone operator in a real Hilbert space H.

$A^{-1}(0) \neq \emptyset$ u(t) → zero of A

- homogenous; $f(t) = 0$

(H1) $A: D(A) \subset H \to H,$ A maximal monotone operator

(H2) $p, q \in L^\infty(R),$ $ess \inf p > 0, q^+ \in L^1(\mathbb{R}_+), q^+ = \max\{q(t), 0\}$

[3] B. Djafari-Rouhani, H. Khatibzadeh, *A note on the strong convergence of solutions to a second order evolution equation*, J. Math. Anal. Appl. 401 (2013) 963–966.