Which Confidence Set Is the Most Robust?

Fredrick Ayivor, Govinda K.C., and
Vladik Kreinovich

University of Texas at El Paso
El Paso, TX 79968, USA

fayivor@miners.utep.edu
gbkc@miners.utep.edu
vladik@utep.edu

1. Formulation of the Problem

- Once we have a probability distribution, then:
  - for each confidence level α,
  - we can form a *confidence set* S, i.e., a set S for which $P(S) = \alpha$.
- There are many such sets, which one should we choose?
- It is reasonable to select S to be connected.
- For each connected confidence set, we can define the degree of robustness by finding out:
  - how much the probability changes
  - if we change the set slightly around some point x on the border ∂S of the set S.

2. Formulation of the Problem (cont-d)

- If we add or subtract a small piece of volume ΔV, then:
  - the change in probability ΔP can be found from the definition of the probability density function (pdf)

$$\rho(x) \stackrel{\text{def}}{=} \lim_{\Delta V \to 0} \frac{\Delta P}{\Delta V},$$

  - as $\Delta P = \rho(x) \cdot \Delta V + o(\Delta V)$.
- Thus, for fixed (and small) ΔV, this change is proportional to $\rho(x)$.
- The value of $\rho(x)$ is, in general, different for different points $x \in \partial S$.
- As a measure of robustness of the given set S, it is reasonable to select the worst-case value

$$r(S) \stackrel{\text{def}}{=} \max_{x \in \partial S} \rho(x).$$

3. Formulation of the Problem (cont-d)

- *Reminder:* as a measure of robustness of the given set S, we select the worst-case value
$$r(S) \stackrel{\text{def}}{=} \max_{x \in \partial S} \rho(x).$$
- How can we select the confidence set with the smallest possible value of $r(S)$?

4. Solution

- We show that the smallest possible value is attained when $\rho(x)$ is constant on ∂S.
- This means that for usual distributions like Gaussian, the corresponding confidence set is

$$S = \{x : \rho(x) \geq \rho_0\} \text{ for some } \rho_0.$$

- For Gaussian distributions, such an ellipsoid is indeed usually selected as a confidence set.

5. Idea of the Proof

- If the function $\rho(x)$ is not constant, then:
  - we can expand S slightly in the vicinity of points x_0 for which $\rho(x_0) \approx \max\limits_x \rho(x)$,
  - thus making the border value of $\rho(x)$ smaller,
  - and at the same time shrink S slightly near all other points $x \in \partial S$,
  - to compensate for the increase in $P(S)$ caused by expanding.
- After this shrinking:
  - the border values of $\rho(x)$ at the corresponding points x will increase,
  - but if we shrink a little bit, these values will still be smaller than $\max\limits_x \rho(x)$.

6. Idea of the Proof (cont-d)

- This way, we will get a new confidence set for which $\max\limits_x \rho(x)$ is slightly smaller.
- Thus, for the set S at which $r(S)$ is the smallest, the values of pdf cannot be non-constant on the border.
- So, for the optimal (most robust) confidence set, we indeed have $\rho(x) = \text{const}$ for all $x \in \partial S$.