What If We Only Know Hurwicz's Optimism-Pessimism Parameter with Interval Uncertainty?

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1. Formulation of the Problem

- In many practical situations, we do not know the exact consequences of each possible action.
- As a result:
  - instead of single utility value u,
  - we can characterize each possible action by the interval $[\underline{u}, \overline{u}]$ of possible utility values.
- In such cases, decision theory recommends an alternative for which the following combination is the largest:

$$\alpha \cdot \overline{u} + (1 - \alpha) \cdot \underline{u} = \underline{u} + \alpha \cdot (\overline{u} - \underline{u}).$$

- The parameter α is known as Hurwicz's *optimism-pessimism* parameter.
- It may be different from different people.

2. Formulation of the Problem (cont-d)

- The value $\alpha = 1$ corresponds to absolute optimists.
- The value $\alpha = 0$ describes a complete pessimist.
- Values between 0 and 1 describe reasonable decision makers.
- The parameter α needs to be determined based on a person's preferences and decisions.
- Often, in different situations, the decisions of the same person correspond to different values α.
- As a result, instead of a single value α, we have the whole range $[\underline{\alpha}, \overline{\alpha}]$ of possible values.
- In this case, how should we make decisions?

3. Our Solution

- For each $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, the interval $[\underline{u}, \overline{u}]$ is equivalent to the value
$$\underline{u} + \alpha \cdot (\overline{u} - \underline{u}).$$
- This expression is monotonic in α.
- So, in general, the original interval the interval $[\underline{u}, \overline{u}]$ is equivalent to the following range of possible values
$$[\underline{u}_1, \overline{u}_1] = [\underline{u} + \underline{\alpha} \cdot (\overline{u} - \underline{u}), \underline{u} + \overline{\alpha} \cdot (\overline{u} - \underline{u})].$$
- Similarly, the interval $[\underline{u}_1, \overline{u}_1]$ is equivalent to
$$[\underline{u}_2, \overline{u}_2] = [\underline{u}_1 + \underline{\alpha} \cdot (\overline{u}_1 - \underline{u}_1), \underline{u}_1 + \overline{\alpha} \cdot (\overline{u}_1 - \underline{u}_1)].$$
- We can repeat this construction again and again:
$$[\underline{u}_{k+1}, \overline{u}_{k+1}] = [\underline{u}_k + \underline{\alpha} \cdot (\overline{u}_k - \underline{u}_k), \underline{u}_k + \overline{\alpha} \cdot (\overline{u}_k - \underline{u}_k)].$$

4. Our Solution (cont-d)

- Reminder:

$$[\underline{u}_{k+1}, \overline{u}_{k+1}] = [\underline{u}_k + \underline{\alpha} \cdot (\overline{u}_k - \underline{u}_k), \underline{u}_k + \overline{\alpha} \cdot (\overline{u}_k - \underline{u}_k)].$$

- At each step, the width of the original intervals decreases by the factor $\overline{\alpha} - \underline{\alpha}$:

$$\overline{u}_{k+1} - \overline{u}_k = (\overline{\alpha} - \underline{\alpha}) \cdot (\overline{u}_k - \underline{u}_k).$$

- Thus, by induction, we conclude that:

$$\overline{u}_k - \underline{u}_k = (\overline{\alpha} - \underline{\alpha})^k \cdot (\overline{u} - \underline{u}).$$

- So,

$$\underline{u}_{k+1} = \underline{u}_k + \underline{\alpha} \cdot (\overline{u}_k - \underline{u}_k) = \underline{u}_k + \underline{\alpha} \cdot (\overline{\alpha} - \underline{\alpha})^k \cdot (\overline{u} - \underline{u}).$$

- Hence,

$$\underline{u}_k = \underline{u} + \underline{\alpha} \cdot (\overline{u} - \underline{u}) + \ldots + \underline{\alpha} \cdot (\overline{u} - \underline{u}) \cdot (\overline{\alpha} - \underline{\alpha})^k.$$

5. Our Solution (cont-d)

- Here, $\underline{u}_k \le \underline{u}_{k+1} \le \overline{u}_{k+1} \le \overline{u}_k$ and $\overline{u}_k - \underline{u}_k \to 0$.
- Thus, in the limit, the intervals $[\underline{u}_k, \overline{u}_k]$ tend to a single point

$$u = \underline{u} + \underline{\alpha} \cdot (\overline{u} - \underline{u}) + \underline{\alpha} \cdot (\overline{u} - \underline{u}) \cdot (\overline{\alpha} - \underline{\alpha}) + \underline{\alpha} \cdot (\overline{u} - \underline{u}) \cdot (\overline{\alpha} - \underline{\alpha})^2 + \ldots$$

- The corresponding geometric progression adds to

$$\underline{u} + \alpha \cdot (\overline{u} - \underline{u}) \text{ for } \alpha = \frac{\underline{\alpha}}{1 - (\overline{\alpha} - \underline{\alpha})}.$$

- This is the desired equivalent value of α for the case when we know α with interval uncertainty.
- This is how we should make decisions in this case.