Experimental Determination of Mechanical Properties Is, In General, NP-Hard – Unless We Measure Everything

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1. Linear Elasticity: a Brief Reminder

- A force applied to a rubber band extends it or curves it.
- In general, a force applied to different parts of an elastic body changes the mutual location of its points.
- Once we know the forces applied at different locations, we can determine the deformations.
- Vice versa, we can determine the forces once we know all the deformations.
- In general, the dependence on forces f_α at different locations α on different displacement ε_β is non-linear.
- However, usually, displacements are small.
- We can ignore terms quadratic or higher order in terms of ε_β.

2. Linear Elasticity (cont-d)

- Thus, we can safely assume that the dependence of each component f_α on ε_β is linear.
- Taking into account that in the absence of forces, there is no displacement, we conclude that $f_\alpha = \sum\limits_\beta K_{\alpha,\beta} \cdot \varepsilon_\beta$.
- The coefficients $K_{\alpha,\beta}$ describe the mechanical properties of the body.
- It is therefore desirable to experimentally determine these coefficients.

3. Ideal Case

- In the ideal case, we measure displacements ε_β and forces f_α at all possible locations.
- Each measurement results in an equation which is linear in terms of the unknowns $K_{\alpha,\beta}$:

$$f_\alpha = \sum_\beta K_{\alpha,\beta} \cdot \varepsilon_\beta$$

- Thus, after performing sufficiently many measurements, we get an easy-to-solve system of linear equations.
- Solving this system enables us to find the values $K_{\alpha,\beta}$.

4. In Practice, We Only Measure Some Values

- In reality, we only measure displacements and forces at some locations.
- So, we know only some values f_α and ε_β.
- Since both $K_{\alpha,\beta}$ and some ε_β are unknown, the corresponding system of equations becomes quadratic.
- After sufficiently many measurements, we may still uniquely determine $K_{\alpha,\beta}$.
- However, the reconstruction is more complex.

5. How Complex: What We Prove

- How complex is the corresponding computational problem?
- In this talk, we prove that the corresponding reconstruction problem is, in general, NP-hard.
- This means that, if – as most computer scientists believe – NP $\neq$ P,
  - no feasible algorithm is possible
  - that would always reconstruct the mechanical properties $K_{\alpha,\beta}$ based on the experimental results.
- We will prove NP-hardness even for the following:
  - given α_0, β_0, and K_0,
  - check whether for some solution, $K_{\alpha_0,\beta_0} = K_0$.

6. Definition

- From the computational viewpoint, the above problem can be formulated as follows.
- Let N be a natural number. This number will be called *the number of experiments.*
- By a *problem of experimentally determining mechanical properties*, we mean the following problem.
  - We know that for every n from 1 to N, we have $f_\alpha^{(n)} = \sum\limits_\beta K_{\alpha,\beta} \cdot \varepsilon_\beta^{(n)}$ for some values $f_\alpha^{(n)}$ and $\varepsilon_\beta^{(n)}$.
  - For each n, we know some of the values $f_\alpha^{(n)}$ and $\varepsilon_\beta^{(n)}$.
  - We need to check if for given α_0, β_0, and K_0, we can have $K_{\alpha_0,\beta_0} = K_0$.

7. Main Result

Proposition. *The problem of experimentally determining mechanical properties is NP-hard.*

8. Proof

- By definition, NP-hard means that all the problems from a certain class NP can be reduced to this problem.
- It is known that the following *subset sum* problem is NP-hard:
  - given $m+1$ natural numbers $s_1, \ldots, s_m, S$,
  - check whether it is possible to find the values $x_i \in \{0,1\}$ for which
$$\sum_{i=1}^{m} s_i \cdot x_i = S.$$
- We check whether there is a subset of the values $s_1, \ldots, s_m$ whose sum is equal to the given value S.
- The subset sum problem is NP-hard.
- This means that every problem from the class NP can be reduced to subset sum.

9. Proof (cont-d)

- So, if we reduce the subset problem to our problem, that would mean, by transitivity of reduction, that
  - every problem from the class NP
  - can be reduced to our problem as well.
- So, our problem is indeed NP-hard.
- Let $s_1, \ldots, s_m, S$ be the values that describe an instance of the subset sum problem.
- Let us reduce it to the following instance of our problem.
- In this instance, we have $2m+1$ variables

$$\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_m, \varepsilon_{m+1}, \ldots, \varepsilon_{2m}.$$

- We also have $m+1$ different values f_α, $\alpha = 0, 1, \ldots, m$.

10. First Series of Experiments

- For each $i = 1, \ldots, m$, we have $\varepsilon_i^{(i)} = 1$, $\varepsilon_{m+i}^{(i)} = -1$, and $\varepsilon_j^{(i)} = 0$ for all $j \neq i$.
- The only value of f_α that we measure in each of these experiments is the value $f_0^{(i)} = 0$; then

$$0 = f_0^{(i)} = \sum_\beta K_{0,\beta} \cdot \varepsilon_\beta^{(i)} = K_{0,i} - K_{0,m+i}.$$

- We conclude that $K_{0,m+i} = K_{0,i}$.

11. Second Series of Experiments

- For each $n = m+i$, we measure $\varepsilon_j^{(m+i)} = 0$ for all $j \neq n$, and we measure $f_0^{(m+i)} = f_i^{(m+i)} = 1$.
- From the corresponding equations, we conclude that $1 = K_{0,m+i} \cdot \varepsilon_{m+i}^{(m+i)}$ and $1 = K_{i,m+i} \cdot \varepsilon_{m+i}^{(m+i)}$.
- We do not know the value $\varepsilon_{m+i}^{(m+i)}$.
- However, we can find it from the first equation and substitute into the second one.
- As a result, we conclude that $K_{0,m+i} = K_{i,m+i}$.
- We know that $K_{0,i} = K_{0,m+i}$, thus $K_{0,i} = K_{i,m+i}$.

12. Third Series of Experiments

- For each i, we measure $\varepsilon_i^{(2m+i)} = 1$, $\varepsilon_j^{(2m+i)} = 0$ for all other j, and we measure $f_i^{(2m+i)} = 1$.
- The corresponding equation implies that $K_{i,i} = 1$.

13. Fourth Series of Experiments

- We measure the values $\varepsilon_{m+i}^{(3m+i)} = -1$ and $\varepsilon_j^{(3m+i)} = 0$ for all $j \neq i, m+i$.
- We also measure the values $f_0^{(3m+i)} = f_i^{(3m+i)} = 0$.
- In this case, we get $K_{0,i} \cdot \varepsilon_i^{(3m+i)} - K_{0,m+i} = 0$ and $K_{i,i} \cdot \varepsilon_i^{(3m+i)} - K_{i,m+i} = 0$.
- Since $K_{i,i} = 1$, we have $\varepsilon_i^{(3m+i)} = K_{i,m+i}$.
- Since $K_{i,m+i} = K_{0,i}$, this implies $\varepsilon_i^{(3m+i)} = K_{0,i}$.
- Let's substitute this expression for $\varepsilon_i^{(3m+i)}$ into
$$K_{0,i} \cdot \varepsilon_i^{(3m+i)} - K_{0,m+i} = 0.$$
- Taking into account that $K_{0,m+i} = K_{0,i}$, we get
$$K_{0,i}^2 - K_{0,i} = 0.$$
- Thus, for each i from 1 to m, we have $K_{0,i} \in \{0, 1\}$.

14. Final (Fifth) Series: A Single Experiment

- We measure $\varepsilon_0^{(4m+1)} = -S$, $\varepsilon_1^{(4m+1)} = s_1, \ldots, \varepsilon_m^{(4m+1)} = s_m$, and $\varepsilon_{m+i}^{(4m+1)} = 0$ for all $i = 1, \ldots, m$.
- We also measure $f_0^{(4m+1)} = 0$.
- We want to check whether it is possible that $K_{0,0} = 1$.
- For $K_{0,0} = 1$, the corresponding equation takes the form $-S + K_{0,1} \cdot s_1 + \ldots + K_{0,m} \cdot s_m = 0$.
- So, $K_{0,1} \cdot s_1 + \ldots + K_{0,m} \cdot s_m = S$ for some $K_{0,i} \in \{0, 1\}$.
- Suppose that the original instance of the subset sum problem has a solution $x_i \in \{0, 1\}$.
- Then the above equality holds for $K_{0,i} = x_i$.

15. Final Series (cont-d)

- Vice versa, suppose that there exist values $K_{0,i} \in \{0, 1\}$ that satisfy the formula

$$K_{0,1} \cdot s_1 + \ldots + K_{0,m} \cdot s_m = S.$$

- Then the values $x_i = K_{0,i}$ solve the original subset sum problem:

$$\sum_{i=1}^{m} s_i \cdot x_i = S.$$

- Thus, we indeed have a reduction – and hence, our problem is indeed NP-hard.

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