Why Geological Regions?

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1. Formulation of the problem

- In many practical problems, we want to describe how the value of some quantity q depends on the 2D or 3D spatial location x.
- This can be the description:
  - of an electromagnetic field or
  - of the state of the atmosphere
- In most such situations, we use smooth (differentiable) functions to describe the dependence $q(x)$.
- However, in geological sciences, the usual description consists of dividing the spatial area into *geological regions*.
- These are zones in each of which the value q is assumed to be constant.
- So why, in geosciences, this different approximating approach is more successful?

2. Our idea

- In general, a natural way to describe an unknown function is to select an orthonormal basis $e_1(x)$, $e_2(x)$, ...
- Then, each function $q(x)$ can be represented as $q(x) = \sum\limits_{i=1}^{\infty} c_i \cdot e_i(x)$, where $c_i = \int q(x) \cdot e_i(x)\,dx$.
- So, with any desired accuracy, we can approximate the function $q(x)$ as $q(x) \approx \sum\limits_{i=1}^{n} c_i \cdot e_i(x)$, for a sufficiently large n.
- In practice, we only know approximate values $\widetilde{q}(x) \approx q(x)$.
- So we get $\widetilde{q}(x) \approx \sum\limits_{i=1}^{n} \widetilde{c}_i \cdot e_i(x)$, where $\widetilde{c}_i = \int \widetilde{q}(x) \cdot e_i(x)\,dx$.
- We want to select the basis $e_i(x)$ for which this approximation is as accurate as possible.
- How can we measure this accuracy?

3. How can we measure approximation accuracy?

- This depends on the application.
- In weather prediction, we are not trying to predict the temperature or the wind speed at every single location in the city.
- Understandably:
  - some areas will be more windy, some less windy,
  - some slightly warmer, some slightly colder.
- What we want to predict is average temperature over some area, average wind speed, etc.
- In such situations, a reasonable measure of accuracy is the usual "average" (mean square) difference $\int (q(x) - \widetilde{q}(x))^2\,dx$.

4. Geosciences are different

- In contrast, in geosciences, we are usually interested in specific locations.
- It is useless to learn that on average, the area contains some oil.
- We want to know where exactly is this oil.
- It makes sense to predict the weather in Southern California in general.
- However, it would be useless to just say that this is a seismic zone.
- We want to know which areas are more vulnerable to future earthquakes.
- We want to make sure that the value $q(x)$ at each location x is accurately approximated, with some accuracy $\varepsilon > 0$.

5. The resulting explanation

- We want to make sure that the sum of the terms $\widetilde{c}_i \cdot e_i(x)$ approximates the sum of the terms $c_i \cdot e_i(x)$.
- It is reasonable to require that each term $\widetilde{c}_i \cdot e_i(x)$ is as close to the corresponding ideal term $c_i \cdot e_i(x)$ as possible.
- In other words, we want to minimize the worst-case approximation error $A \stackrel{\text{def}}{=} \max\limits_{x,q(x),\widetilde{q}(x)} |\widetilde{c}_i \cdot e_i(x) - c_i \cdot e_i(x)|$; here:
  - we denoted $c_i = \int q(x) \cdot e_i(x)\,dx$ and $\widetilde{c}_i = \int \widetilde{q}(x) \cdot e_i(x)\,dx$, and
  - the maximum is taken over all the functions $q(x)$ and $\widetilde{q}(x)$ for which $|\widetilde{q}(x) - q(x)| \le \varepsilon$ for all x.
- It turns out that the smallest value of this worst-case approximation error A is attained when the function $e_i(x)$ is piece-wise constant.
- This explains why such an approximation – corresponding to geological regions – is indeed very effective in geosciences.

6. Proof

- We want to minimize $A \stackrel{\text{def}}{=} \max\limits_{x,q(x),\widetilde{q}(x)} |\widetilde{c}_i \cdot e_i(x) - c_i \cdot e_i(x)|$.
- Here, $\widetilde{c}_i \cdot e_i(x) - c_i \cdot e_i(x) = \Delta c_i \cdot e_i(x)$, where

$$\Delta c_i \stackrel{\text{def}}{=} \widetilde{c}_i - c_i = \int \Delta q(x) \cdot e_i(x)\, dx \text{ and } \Delta q(x) \stackrel{\text{def}}{=} \widetilde{q}(x) - q(x).$$

- Thus, $A = \max\limits_{x,\Delta q(x)} |\Delta c_i \cdot e_i(x)| = \max\limits_{x,\Delta q(x)} \left(|\Delta c_i| \cdot |e_i(x)|\right)$.
- The only condition of $\Delta q(x)$ is that $|\Delta q(x)| \le \varepsilon$.
- The maximized expression $|\Delta c_i| \cdot |e_i(x)|$ is the product of two terms:
  - the term $|\Delta c_i|$ only depends on $\Delta q(x)$, and
  - the term $|e_i(x)|$ only depends on x.
- Thus, $A = \left(\max\limits_{\Delta q(x)} |\Delta c_i|\right) \cdot \left(\max\limits_{y} |e_i(y)|\right)$.
- The largest value of the sum $\Delta c_i = \int \Delta q(x) \cdot e_i(x)\, dx$ is attained when each term $\Delta q(x) \cdot e_i(x)$ is the largest.

7. Proof (cont-d)

- When $e_i(x) \geq 0$, maximum is attained when $\Delta q(x)$ is the largest $\Delta q(x) = \varepsilon$, then $\Delta q(x) \cdot e_i(x) = \varepsilon \cdot e_i(x)$.
- When $e_i(x) \leq 0$, maximum is attained when $\Delta q(x)$ is the smallest $\Delta q(x) = -\varepsilon$, then $\Delta q(x) \cdot e_i(x) = -\varepsilon \cdot e_i(x)$.
- In both cases, the largest value is equal to $\varepsilon \cdot |e_i(x)|$; thus:

$$\max_{\Delta q(x)} |\Delta c_i| = \max_{\Delta q(x)} \left| \int \Delta q(x) \cdot e_i(x)\, dx \right| = \int \varepsilon \cdot |e_i(x)|\, dx = \varepsilon \cdot \int |e_i(x)|\, dx.$$

- So, $A = \varepsilon \cdot \left(\int |e_i(x)|\, dx \right) \cdot \max\limits_y |e_i(y)|$.
- Minimizing A is equivalent to minimizing

$$J \stackrel{\text{def}}{=} \frac{A}{\varepsilon} = \left(\int |e_i(x)|\, dx \right) \cdot \max_y |e_i(y)|.$$

- The functions $e_i(x)$ are orthonormal, so

$$\int e_i^2(x)\, dx = \int |e_i(x)| \cdot |e_i(x)|\, dx = 1.$$

8. Proof (cont-d)

- For each x, we have $|e_i(x)| \le \max\limits_y |e_i(y)|$; so:

$$1 = \int |e_i(x)| \cdot |e_i(x)|\,dx \le \int \left(\max_y |e_i(y)|\right) \cdot |e_i(x)|\,dx =$$

$$\max_y |e_i(y)| \cdot \int |e_i(x)|\,dx = J.$$

- If at least for one x, we have $|e_i(x)| \cdot |e_i(x)| < \left(\max\limits_y |e_i(y)|\right) \cdot |e_i(x)|$, then $1 < J$.
- The smallest possible value $J = 1$ is therefore attained if for all x, we have:

$$|e_i(x)| \cdot |e_i(x)| = \left(\max_y |e_i(y)|\right) \cdot |e_i(x)|.$$

- If $|e_i(x)| = 0$, this equality is always satisfied.

9. Proof (cont-d)

- If $|e_i(x)| \neq 0$, then we can divide both side of this equality by $|e_i(x)|$ and get $|e_i(x)| = \max\limits_y |e_i(y)|$.
- So, for each x, the value of $e_i(x)$ is:
  - either equal to 0,
  - or equal to $\pm \max\limits_y |e_i(y)|$.
- Thus, the optimal function $e_i(x)$ is indeed piecewise-constant.

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