Decision Making Under Uncertainty: Case When We Only Know an Upper Bound or a Lower Bound

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1. Formulation of the problem

- In investment:
  - when a person knows the exact monetary consequence of each action,
  - he/she naturally selects an action with the largest possible gain.
- In practice, we usually know the consequences only with some uncertainty.
- For example, instead of the exact gain value, the whole set S of different possible gain values are consistent with our knowledge.
- How should we then make a decision?
- What is the equivalent price $v(S)$ that we are willing to pay to participate in the corresponding action?

2. Formulation of the problem (cont-d)

- For example, we may know the lower bound a and the upper bound on the gain.
- In this case, the set S is the interval $[a, b]$.
- Alternatively, we may know:
  - only the lower bound, in which case $S = [a, \infty)$ or
  - only the upper bound, in which case $S = (-\infty, b]$.

3. How this problem is solved if we know both bounds

- Suppose that we are willing to pay $v(S)$ for the set S.
- Then for set S *and* a fixed amount c, we are willing to pay $v(S) + c$.
- In this joint offer, the set of possible outcomes is

$$S + c \stackrel{\text{def}}{=} \{s + c : s \in S\}.$$

- Thus, we should have $v(S + c) = v(S) + c$.
- This is called *shift-invariance*.
- Another idea is that the transformation $S \mapsto v(S)$ should not depend on the choice of the monetary unit:
  - if we select pesos instead of dollars,
  - we should get the same equivalent value.

4. What if we know both bounds (cont-d)

- In precise terms, this means $v(\lambda \cdot S) = \lambda \cdot v(S)$, where

$$\lambda \cdot S \stackrel{\text{def}}{=} \{\lambda \cdot s : s \in S\}.$$

- This property is known as *scale-invariance.*
- The third idea is that:
  - participation in two independence actions, with sets S_1 and S_2, is equivalent to
  - participation in a single action with the result

$$S_1 + S_2 = \{s_1 + s_2 : s_1 \in S_1 \,\&\, s_2 \in S_2\}.$$

- These are two ways of representing the same situation.
- So we should have $v(S_1 + S_2) = v(S_1) + v(S_2)$.
- This property is known as *additivity*.

5. What if we know both bounds (cont-d)

- For interval uncertainty, additivity implies Hurwicz formula $v([a,b]) = \alpha \cdot b + (1-\alpha) \cdot a$ for some $\alpha \in [0,1]$.
- The same formula emerges if we assume shift- and scale-invariance.

6. Case of infinite intervals: new results

- It turns out that additivity implies that $f(a) \stackrel{\text{def}}{=} v([a,\infty)) = k \cdot a$ for some $k \geq 1$.
- Indeed, additivity means that $f(a+b) = f(a) + f(b)$, for $f(a) \geq a$.
- It is known that this functional equation implies $f(a) = k \cdot a$.
- A similar formula emerges if we assume scale-invariance.
- Indeed, scale-invariance means $f(\lambda \cdot a) = \lambda \cdot f(a)$ for all $\lambda > 0$ and a.
- In particular, for $a = 1$, we get $f(\lambda) = k_+ \cdot \lambda$, where $k_+ \stackrel{\text{def}}{=} f(1)$.
- For $a = -1$, we similarly get $f(-\lambda) = k_- \cdot \lambda$, i.e., $f(x) = (-k_-) \cdot x$.
- If we assume shift-invariance, then we get $f(a) = v([a,\infty)) = a + a_0$ for some $a_0 \geq 0$.
- Indeed, here, shift-invariance means $f(a+c) = f(a) + c$.
- For $a = 0$, we get $f(c) = c + a_0$, where $a_0 \stackrel{\text{def}}{=} f(0)$.

7. Case of infinite intervals: new results (cont-d)

- Similar results emerge for $S = (-\infty, b]$.
- Here, additivity implies that $g(b) \stackrel{\text{def}}{=} v((-\infty, b]) = k \cdot b$ for some $0 \le k \le 1$.
- If we assume scale-invariance, then:
  - we get $g(b) = k_+ \cdot b$ for $b \ge 0$, and
  - we get $g(b) = k_- \cdot b$ for $b \le 0$.
- If we assume shift-invariance, then we get $v((-\infty, b]) = b - b_0$ for some $b_0 \ge 0$.

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