Commonsense “And”-Operations

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1. Why "and"-operations

- In many practical applications, a certain effect appears if several conditions $C_1, C_2, \ldots$ are satisfied.
- For each of these conditions C_i, we can elicit, from the experts, the degree $d_i \in [0, 1]$ to which this condition is satisfied.
- However, there are many possible conditions.
- It is not possible to extract, from the experts, a degree to which each possible "and"-combination $C_1 \,\&\, C_2 \,\&\, \ldots$ is satisfied.
- Thus, we need to be able:
  - given degrees of confidence a and b in statements A and B,
  - to estimate the degree to which the "and"-combination $A \,\&\, B$ is satisfied.
- This estimate is denoted by $f_\&(a, b)$.
- The algorithm for computing this estimate is known as an "and"-operation or, for historical reason, a t-norm.

2. How usual "and"-operations are obtained

- In some situations, about each of the combined statements, we are absolute certain
  - either that this statement is true
  - or that this statement is false.
- Then, the "and"-operation should return the true value of the corresponding "and"-statement.
- So we should have $f_\&(0,0) = f_\&(0,1) = f_\&(1,0) = 0$ and $f_\&(1,1) = 1$.
- We want to extend these values to all possible combinations of $a \in [0,1]$ and $b \in [0,1]$.
- A reasonable idea is to use linear interpolation over each variable, i.e., to assume that:
  - for every a, the mapping $b \mapsto f_\&(a,b)$ is linear, and
  - for every b, the mapping $a \mapsto f_\&(a,b)$ is linear.

3. How usual "and"-operations are obtained (cont-d)

- As a result, we conclude that the desired function is bilinear.
- In precise terms, it has the form $f_\&(a,b)=c_0+c_a\cdot a+c_b\cdot b+c_{ab}\cdot a\cdot b$ for some coefficients c_i.
- Taking into account the above conditions for $a,b\in\{0,1\}$, we conclude that $f_\&(a,b)=a\cdot b$.
- This is indeed one of the most frequently used "and"-operations.
- Similarly, linear interpolation enables us to similarly determine that:
  - an appropriate "or"-operation (historically also known as t-conorm)
  - has the form $f_\vee(a,b)=a+b-a\cdot b$.

4. Need to go beyond the usual "and"-operations

- In some cases, when we say "and", we mean exactly the logical "and".
- All conditions must be absolutely satisfied.
- However, in many practical problems, "and" is "softer" than that.
- For example, if you ask a person who is planning to buy a house what house he/she wants, the person will say:
  - not too far away
  - *and* spacey
  - *and* not very expensive
  - *and* reasonably well thermo-isolated
  - *and* in a nice neighborhood, etc.

5. Need to go beyond the usual "and"-operations (cont-d)

- However, this "and" does not mean literal "and".
- If this person finds a house that satisfied most of these conditions, he/she will gladly buy it.
- How can we describe such commonsense "and"-operations?

6. Our solution

- In this talk, we consider the case when we only have two conditions.
- For a commonsense "and"-operation $F_{\&}(a, b)$, it is reasonable to still have $F_{\&}(0, 0) = 0$ and $F_{\&}(1, 1) = 1$.
- However:
  - if only one of the conditions A and B is satisfied,
  - then the statement $A \& B$ should also be to some extent true.
- In other words, we should have $F_{\&}(0, 1) = F_{\&}(1, 0) = \alpha$ for some small $\alpha > 0$.
- In this case, we get $F_{\&}(a, b) = \alpha \cdot (a + b) + (1 - 2\alpha) \cdot a \cdot b$.
- Equivalently,

  $$F_{\&}(a, b) = (1-\alpha)\cdot a\cdot b+\alpha\cdot(a+b-a\cdot b) = (1-\alpha)\cdot f_{\&}(a, b)+\alpha\cdot f_{\vee}(a, b).$$
- In other words, this operation is a convex combination of the usual "and"- and "or"-operations.

7. Discussion

- The usual "and"-operation is associative.
- Thus, we can define $f_{\&}(a, b, c)$ as, e.g., $f_{\&}(a, f_{\&}(b, c))$ or as $f_{\&}(f_{\&}(a, b), c)$ – and the result will not change.
- In contrast, the commonsense "and"-operation is not associative.
- With the commonsense "and"-operation, we will have two different results.
- So, e.g., for three inputs, we get a more general formula

  $$F_{\&}(a, b, c) = \alpha \cdot (a + b + c) + \beta \cdot (a \cdot b + b \cdot c + a \cdot c) + (1 - 3\alpha - 3\beta) \cdot a \cdot b \cdot c.$$

8. Reference

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