Word Representation: Theoretical Explanation of an Empirical Formula

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1. Formulation of the Problem

- Computer representation of natural-language words should reflect how related these words are.
- This relation can be described, e.g., by the number $X_{ij}$ of times when word $i$ appears in the context of word $j$.
- It is desirable to find out how such characteristics depend on the properties of the words $i$ and $j$.
- This way we will be able to predict, e.g., $X_{ij}$ for pairs $(i, j)$ for which we do not know $X_{ij}$.
- At present, these characteristics are determined by training a neural network.
- There is also a reasonably good approximate analytical formula
  \[ \ln(X_{ij}) \approx b_i + \tilde{b}_j + w_i \cdot \tilde{w}_j. \]
- Here, $b_i$ and $\tilde{b}_j$ are numbers, $w_i$ and $\tilde{w}_j$ are vectors, and $a \cdot b$ is dot (scalar) product.
2. Formulation of the Problem (cont-d)

- The values $b_i, \tilde{b}_j, w_i$ and $\tilde{w}_j$ can be found by using the Least Squares method:
  \[
  J \overset{\text{def}}{=} \sum_{i,j} f(X_{ij}) \cdot (b_i + \tilde{b}_j + w_i \cdot \tilde{w}_j - \ln(X_{ij}))^2 \rightarrow \min.
  \]

- The efficiency of this method depends on the appropriate choice of the weight function $f(X)$.

- Empirical data shows that the most efficient is power law $f(X) = X^a$.

- How can we explain this empirical fact?
3. Our Explanation

- The values $X_{ij}$ depend on the size of the corpus.
- If we consider twice smaller corpus, each value $X_{ij}$ will decrease approximately by half.
- In general, if we consider a $\lambda$ times larger corpus, we will get new values which are close to $\lambda \cdot X_{ij}$.
- The word representation should depend only on the words, not on corpus size.
- So, the resulting representation should not change if we replace $X_{ij}$ with $\lambda \cdot X_{ij}$.
- Of course, if we replace $X_{ij}$ with $\lambda \cdot X_{ij}$, the weights will change.
- However, this does not necessarily mean that the resulting representations will change.
4. Our Explanation (cont-d)

- Namely, for any $c > 0$, optimizing any function $J$ is equivalent to optimizing the function $c \cdot J$.

- Example: the richest person on Earth is the richest whether we count his richness in dollars on in pesos.

- If we replace $f(X)$ with $c \cdot f(X)$, we will get $c \cdot J$ instead of $J$, so we will get the same representations $w_i$.

- Thus, we can have $f(\lambda \cdot X) = c \cdot f(X)$, and the resulting representation will be the same.

- So, the invariance with respect to corpus size means that for every $\lambda > 0$, there exists $c$ depending on $\lambda$ for which

$$f(\lambda \cdot X) = c(\lambda) \cdot f(X).$$
5. Our Explanation (cont-d)

- It is known that all measurable solutions to the functional equation
  \( f(\lambda \cdot X) = c(\lambda) \cdot f(X) \) are power laws
  \( f(X) = A \cdot X^a \).

- As we have mentioned, the use of such weights is equivalent to using
  the weights \( f(X) = X^a \).

- This explains why the power law weights work the best.

- Power law weights are the only ones for which the resulting represen-
  tation does not depend on the corpus size.
6. How to Prove the Result about the Functional Equation

- This result is easy to prove when the function \( f(X) \) is differentiable.
- Suppose that \( f(\lambda \cdot X) = c(\lambda) \cdot f(X) \).
- If we differentiate both sides with respect to \( \lambda \), we get
  \[
  X \cdot f'(\lambda \cdot X) = c'(\lambda) \cdot f(X).
  \]
- In particular, for \( \lambda = 1 \), we get \( X \cdot f'(X) = a \cdot f(X) \), where \( a \overset{\text{def}}{=} c'(1) \), so \( X \cdot \frac{df}{dX} = a \cdot f \).
- We can separate the variables if we multiply both sides by \( \frac{dX}{X \cdot f} \), then we get \( \frac{df}{f} = a \cdot \frac{dX}{X} \).
- Integrating both sides of this equality, we get \( \ln(f) = a \cdot \ln(X) + C \).
- By applying \( \exp(x) \) to both sides, we get
  \[
  f(X) = \exp(a \cdot \ln(X) + C) = A \cdot X^a, \quad \text{where} \quad A \overset{\text{def}}{=} e^C.
  \]
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