Need for Optimal Distributed Measurement of Cumulative Quantities Explains the Ubiquity of Absolute and Relative Error Components

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1. Need for distributed measurements

- In many practical situations, we are interested in estimating the value x of a cumulative quantity; e.g, we want to estimate:
  - the overall amount of oil in a given area,
  - the overall amount of CO_2 emissions, etc.
- Measuring instruments usually measure quantities in a given location.
- Thus, they measure local values $x_1, \ldots, x_n$ that together form the desired value $x = x_1 + \ldots + x_n$.
- So, a natural way to produce an estimate $\widetilde{x}$ for x is:
  - to place measuring instruments at several locations,
  - to measure the values x_i in these locations, and
  - to add up the results $\widetilde{x}_1 + \ldots + \widetilde{x}_n$ of these measurement:

$$\widetilde{x} = \widetilde{x}_1 + \ldots + \widetilde{x}_n.$$

2. Need for optimal planning

- Usually, we want to reach a certain estimation accuracy.
- To achieve this accuracy, we need to plan how accurate the deployed measurement instruments should be.
- Use of accurate measuring instruments is often very expensive, while budgets are usually limited.
- It is therefore desirable to come up with the deployment plan that would achieve the desired overall accuracy within the minimal cost.
- This implies, in particular, that the resulting estimate should not be more accurate than needed.
- Indeed, this would mean that we could use less accurate (and thus, cheaper) measuring instruments.

3. Need for optimal planning (cont-d)

- In this talk, we provide:
  - a condition under which such optimal planning is possible, and
  - the corresponding optimal planning algorithm.
- The resulting condition explains why usually, measuring instruments are characterized by their absolute and relative accuracy.

4. How we can describe measurement accuracy

- Measurements are never absolutely accurate.
- The measurement result $\widetilde{x}_i$ is, in general, different from the actual (unknown) value x_i of the corresponding quantity.
- In other words, the difference $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$ is, in general, different from 0.
- This difference is known as the *measurement error.*
- For each measuring instrument, we should know how large the measurement error can be.
- In precise terms, we need to know an upper bound Δ on the absolute value $|\Delta x_i|$ of the measurement error.
- This upper bound should be provided by the manufacturer of the measuring instrument.

5. How we can describe measurement accuracy (cont-d)

- Indeed, if no such upper bound is known, this means that:
  - whatever the reading of the measuring instrument,
  - the actual value can be as far away from this reading as possible.
- So we get no information whatsoever about the actual value – in this case, this is not a measuring instrument, it is a wild guess.
- Ideally:
  - in addition to knowing that the measurement error Δx_i is somewhere in the interval $[-\Delta, \Delta]$,
  - it is desirable to know how probable are different values from this interval,
  - i.e., what is the probability distribution on the measurement error.
- Sometimes, we know this probability distribution.
- However, in many practical situations, we don't know it, and the upper bound is all we know.

6. How we can describe measurement accuracy (cont-d)

- So, we will consider this upper bound as the measure of the instrument's accuracy.
- This upper bound Δ may depend on the measured value.
- For example, suppose that we are measuring current in the range from 1 mA to 1 A.
- Then, it is relatively easy to maintain accuracy of 0.1 mA when the actual current is 1 mA.
- This means measuring with one correct decimal digit.
- We can get values $0.813\ldots$, $0.825\ldots$
- However, since the measurement accuracy is 0.1, this means that these measurement results may correspond to the same actual value.
- In other words, whatever the measuring instrument shows, only one digit is meaningful and significant.

7. How we can describe measurement accuracy (cont-d)

- All the other digits may be caused by measurement errors.
- But can we maintain the same accuracy of 0.1 mA when we measure currents close to 1 A?
- This would mean that we need to distinguish between values 0.94651 A = 946.51 mA and 0.94637 A = 946.37 mA.
- Indeed, the difference between these two values is larger than 0.1 mA.
- This would mean that we require that in the measurement result, we should have not one, but four significant digits.
- This would require much more accurate measurements.
- Because of this, we will explicitly take into account that the accuracy Δ depends on the measured value: $\Delta = \Delta(x)$.
- Usually, small changes in x lead to only small changes in the accuracy.
- So, we can safely assume that the dependence $\Delta(x)$ is smooth.

8. What we want

- We want to estimate the desired cumulative value x with some accuracy δ.
- In other words, we want to make sure that the difference between our estimate $\widetilde{x}$ and the actual value x does not exceed δ: $|\widetilde{x} - x| \le \delta$.
- The cumulative value is estimated based on n measurement results.
- As we have mentioned, the accuracy that we can achieve in each measurement, in general, depends on the measured value.
- The larger the value of the measured quantity, the more difficult it is to maintain the corresponding accuracy.
- It is therefore reasonable to conclude that:
  - whatever measuring instruments we use to measure each value x_i,
  - it will be more difficult to estimate the larger cumulative value x with the same accuracy.

9. What we want (cont-d)

- Thus, it makes sense to require that the desired accuracy δ should also depend on the value that we want to estimate $\delta = \delta(x)$.
- The larger the value x, the larger the uncertainty $\delta(x)$ that we can achieve.
- So, our problem takes the following form:
- We want to be able to estimate the cumulative value x with given accuracy $\delta(x)$.
- In other words, we are given a function $\delta(x)$ and we want to estimate the cumulative value with this accuracy
- We want to find the measuring instruments:
  - that would guarantee this estimation accuracy, and
  - that would be optimal for this task, i.e., that would not provide better accuracy than needed.

10. Let us describe what we want in precise terms

- Let us analyze what estimation accuracy we can achieve if we use:
  - for each of n measurements,
  - the measuring instrument characterized by the accuracy $\Delta(x)$.
- Let $\widetilde{x}_i$ be the i-th measurement result.
- Then, the actual value x_i of the corresponding quantity is located somewhere on the interval $[\widetilde{x}_i - \Delta(x_i), \widetilde{x}_i + \Delta(x_i)]$.
- The smallest possible value is $\widetilde{x}_i - \Delta(x_i)$.
- The largest possible value is $\widetilde{x}_i + \Delta(x_i)$.
- When we add the measurement results, we get the estimate $\widetilde{x} = \widetilde{x}_1 + \ldots + \widetilde{x}_n$ for the desired quantity x.
- What are the possible values of this quantity?

11. Let us describe what we want in precise terms (cont-d)

- The sum $x = x_1 + \ldots + x_n$ attains its smallest value if all values x_i are the smallest, i.e., when

  $$x = (\widetilde{x}_1 - \Delta(x_1)) + \ldots + (\widetilde{x}_n - \Delta(x_n)) = (\widetilde{x}_1 + \ldots + \widetilde{x}_n) - (\Delta(x_1) + \ldots + \Delta(x_n)).$$

- In this case, $x = \widetilde{x} - (\Delta(x_1) + \ldots + \Delta(x_n))$.
- Similarly, the sum $x = x_1 + \ldots + x_n$ attains its largest value if all values x_i are the largest, i.e., when

  $$x = (\widetilde{x}_1 + \Delta(x_1)) + \ldots + (\widetilde{x}_n + \Delta(x_n)) = (\widetilde{x}_1 + \ldots + \widetilde{x}_n) + (\Delta(x_1) + \ldots + \Delta(x_n)).$$

- In this case, $x = \widetilde{x} + (\Delta(x_1) + \ldots + \Delta(x_n))$.
- Thus, all we can conclude about the value x is that this value belongs to the interval

  $$[\widetilde{x} - (\Delta(x_1) + \ldots + \Delta(x_n)), \widetilde{x} + (\Delta(x_1) + \ldots + \Delta(x_n))].$$

- This means that we get an estimate of x with the accuracy

  $$\Delta(x_1) + \ldots + \Delta(x_n).$$

12. Let us describe what we want in precise terms (cont-d)

- Our objective is to make sure that this is exactly the desired accuracy $\delta(x)$.
- In other words, we want to make sure that whenever $x = x_1 + x_2 + \ldots + x_n$, we should have $\delta(x) = \Delta(x_1) + \Delta(x_2) + \ldots + \Delta(x_n)$.
- Substituting $x = x_1 + x_2 + \ldots + x_n$ into this formula, we get

$$\delta(x_1 + x_2 + \ldots + x_n) = \Delta(x_1) + \Delta(x_2) + \ldots + \Delta(x_n).$$

- We do not know a priori what will be the values x_i.
- We want to maintain the desired accuracy $\delta(x)$ – and make sure that we do not get more accuracy.
- So, we should make sure that the equality be satisfied for all possible values $x_1, x_2, \ldots, x_n$.

13. Let us describe what we want in precise terms (cont-d)

- In these terms, the problem takes the following form:
  - for which functions $\delta(x)$ is it possible to have a function $\Delta(x)$ for which the above equality is satisfied? and
  - how can we find this function $\Delta(x)$ – that describes the corresponding measuring instrument?
- This is the problem that we solve in this talk.

14. When Is Optimal Distributive Measurement of Cumulative Quantities Possible?

- We assumed that the function $\Delta(x)$ is smooth, i.e., differentiable.
- Thus, the sum $\delta(x)$ of such functions is differentiable too.
- Since both functions $\Delta(x)$ and $\delta(x)$ are differentiable, we can differentiate both sides of the above equality with respect to x_1.
- The terms $\Delta(x_2), \ldots, \Delta(x_n)$ do not depend on x_1 at all, so their derivative with respect to x_1 is 0.
- Thus, the resulting formula takes the form

$$\delta'(x_1 + x_2 + \ldots + x_n) = \Delta'(x_1).$$

- Here, as usual, δ' and Δ' denote the derivatives of the corresponding functions.
- The new equality holds for all possible values $x_2, \ldots, x_n$.
- For every real number x_0, we can take, e.g., $x_2 = x_0 - x_1$ and $x_3 = \ldots = x_n = 0$, then we will have $x_1 + x_2 + \ldots + x_n = x_0$.

15. When Is Optimal Distributive Measurement of Cumulative Quantities Possible (cont-d)

- So, the equality takes the form $\delta'(x_0) = \Delta'(x_1)$.
- The right-hand side does not depend on x_0, which means that the derivative $\delta'(x_0)$ is a constant not depending on x_0 either.
- The only functions whose derivative is a constant are linear functions.
- So we conclude that the dependence $\delta(x)$ is linear: $\delta(x) = a + b \cdot x$ for some constants a and b.
- Interestingly, this fits well with the usual description of measurement error, as consisting of two components:
  - the absolute error component a that does not depend on x at all,
  - and the relative error component.

16. When Is Optimal Distributive Measurement of Cumulative Quantities Possible (cont-d)

- For relative error, the bound on the measurement error is a certain percentage of the actual value x.
- So, it has the form $b \cdot x$ for some constant b.
- Example: 10% accuracy means $b = 0.1$. Thus, our result explains this usual description.

17. What Measuring Instrument Should We Select to Get the Optimal Distributive Measurement?

- Now we know for what desired accuracy $\delta(x)$, we can have the optimal distributive measurement of a cumulative quantity.
- The natural next question is:
  - given one of such functions $\delta(x)$,
  - what measuring instrument – i.e., what function $\Delta(x)$ – should we select for this optimal measurement?
- To answer this question, we can take $x_1 = \ldots = x_n$.
- In this case, $\Delta(x_1) = \ldots = \Delta(x_n)$, so the above equality takes the form $\delta(n \cdot x_1) = n \cdot \Delta(x_1)$.
- We know that $\delta(x) = a + b \cdot x$, so $a + b \cdot n \cdot x_1 = n \cdot \Delta(x_1)$.

18. What Measuring Instrument Should We Select to Get the Optimal Distributive Measurement (cont-d)

- If we divide both sides of this equality by n, and rename x_1 into x, we get the desired expression for $\Delta(x)$: $\Delta(x) = \dfrac{a}{n} + b \cdot x$.
- In other words:
  - the bound on the relative error component of each measuring instrument should be the same as for the cumulative quantity;
  - the bound on the absolute error component should be n times smaller than for the cumulative quantity.

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