

Can We Uniquely Reconstruct a Function from Measurements in Which Input Is Known with Interval Uncertainty?

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1. Formulation of the problem

- In many practical situations, we know that the value of the quantity y is uniquely determined by the value of a related quantity x .
- In mathematical terms, this means that y is a function of x : $y = f(x)$.
- However, we often do not know the exact expression for this dependence.
- In such situations, to find this dependence, we set up a value x and measure the corresponding value y .
- For example, to find possible deviations from Ohm's law, we set up different voltages x and measure the corresponding current y .

2. Formulation of the problem (cont-d)

- In the ideal case, when we can set up x with absolute accuracy and measure y with absolute accuracy, we get many pairs $(x, f(x))$. So:
 - after using values x which are densely distributed in the given range,
 - we can practically uniquely determine all the values of the desired function $f(x)$.
- In practice, often, while we can measure y accurately, we can only measure x with some accuracy δ .
- So, based on the measurement result \tilde{x} , we can only guarantee that the actual value x is in the interval $[\tilde{x} - \delta, \tilde{x} + \delta]$ of width 2δ .
- In principle, for any x from this interval, we can have measurement result \tilde{x} .

3. Formulation of the problem (cont-d)

- So, if we repeat measurement many times, we will encounter situations covering many such x 's.
- For each such x , we measure the value $f(x)$.
- So, after all these measurements, we get, in effect, the set of all possible values y corresponding to different values $x \in [\tilde{x} - \delta, \tilde{x} + \delta]$:

$$[\underline{y}(\tilde{x}), \overline{y}(\tilde{x})] \stackrel{\text{def}}{=} \{f(x) : x \in [\tilde{x} - \delta, \tilde{x} + \delta]\}$$

- A natural question is: for a fixed accuracy δ , can we uniquely reconstruct the function $f(x)$ based on these ranges?
- And if we cannot always reconstruct the desired function uniquely, when *can* we do it?

4. Sometimes we can: case of monotonicity

- When the function $f(x)$ is (non-strictly) increasing, then

$$\underline{y}(\tilde{x}) = f(\tilde{x} - \delta) \text{ and } \bar{y}(\tilde{x}) = f(\tilde{x} + \delta).$$

- Thus, by considering appropriate values \tilde{x} , we can find the value $f(x)$ for all x .
- Similarly, when the function $f(x)$ is non-strictly decreasing.
- In this case, $\underline{y}(\tilde{x}) = f(\tilde{x} + \delta)$ and $\bar{y}(\tilde{x}) = f(\tilde{x} - \delta)$.
- So here also, by considering appropriate values \tilde{x} , we can find the value $f(x)$ for all x .
- A more general case is when the range of the function consists of several intervals $[\underline{X}, \bar{X}]$:
 - on each of which $f(x)$ is either increasing or decreasing, and
 - the width of each such interval is at least 4δ .

5. Sometimes we can: case of monotonicity (cont-d)

- Then, all the values from $f(x)$ for $x \in [\underline{X}, \overline{X} - 2\delta]$ can be obtained as $f(\tilde{x} - \delta)$ for $\tilde{x} = x + \delta$.
- For this \tilde{x} , we have $[\tilde{x} - \delta, \tilde{x} + \delta] \subseteq [\underline{X}, \overline{X}]$.
- So $f(x)$ is monotonic on $[\tilde{x} - \delta, \tilde{x} + \delta]$ and $f(\tilde{x} - \delta)$ can be computed.
- Similarly, all the values from $f(x)$ for $x \in [\underline{X} + 2\delta, \overline{X}]$ can be obtained as $f(\tilde{x} + \delta)$ for $\tilde{x} = x - \delta$.
- For this \tilde{x} , we have $[\tilde{x} - \delta, \tilde{x} + \delta] \subseteq [\underline{X}, \overline{X}]$.
- So $f(x)$ is monotonic on $[\tilde{x} - \delta, \tilde{x} + \delta]$ and $f(\tilde{x} + \delta)$ can be computed.
- Since $\overline{X} - \underline{X} \geq 4\delta$, we have $\underline{X} + 2\delta \leq \overline{X} - 2\delta$.
- Thus, every point from the range $[\underline{X}, \overline{X}]$ is so covered.

6. Sometimes, we cannot

- If the range of monotonicity is too narrow, we may not be able to reconstruct the function uniquely.
- For example, if $f(x) = \sin(x)$, then for $\delta = \pi$ each interval $[\tilde{x} - \delta, \tilde{x} + \delta]$ contains the full period of the function.
- So, we have $[\underline{y}(\tilde{x}), \bar{y}(\tilde{x})] = \{f(x) : x \in [\tilde{x} - \delta, \tilde{x} + \delta]\} = [-1, 1]$ for all \tilde{x} .
- The same measurements will happen if $f(x) = \cos(x)$, so the function cannot be uniquely determined.

7. Remaining open question

- It is not clear which functions can be uniquely reconstructed and which cannot.
- Our hypothesis is that the $\geq 4\delta$ width of all monotonicity ranges is not only sufficient but also necessary for reconstruction uniqueness.

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