Multi-Objective Optimization: Linear Combinations Do Not Cover Pareto Set, So What Does?

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1. Formulation of the problem

- In some practical situation, we have a clear objective: to optimize a known objective function $f(x)$ – e.g., profit for companies.
- Many effective algorithms are known for solving such well-defined optimization problems.
- Often, however, the problems are not so well-defined: there are several different objective functions $f_1(x), \ldots, f_n(x)$.
- Usually, we calibrate them so that the status quo state s (when we do not make any decision) corresponds to $f_i(s) = 0$.
- This way, we are only looking for alternatives x for which $f_i(x) \geq 0$ for all i.
- In such situations, we do not want alternatives x which are *dominated* by others, i.e., for which, for some y:
  - we have $f_i(x) \leq f_i(y)$ for all i and
  - we have $f_j(x) < f_j(y)$ for some j.

2. Formulation of the problem (cont-d)

- We thus want to generate the set P of all non-dominated alternatives, so that a human decision maker can make a final decision.
- This problem is known as *multi-objective optimization.*
- The set P is known as the *Pareto set.*
- There are many effective algorithms for solving traditional optimization problem.
- So, a natural idea is to reduce multi-objective optimization to several optimization ones.
- Namely, we select a continuous function $F(v_1, \ldots, v_n)$ which:
  - is (non-strictly) increasing relative to each of its inputs and
  - which does not change under permutations.
- Then, we optimize functions $F(c_1 \cdot f_1(x), \ldots, c_n \cdot f_n(x))$ corresponding to different $c_i \geq 0$.

3. Formulation of the problem (cont-d)

- It is also reasonable to require that:
  - the function F is *homogeneous*, i.e., that for all c and all v_i, we have:
  $$F(c \cdot v_1, \ldots, c \cdot v_n) = c \cdot F(v_1, \ldots, v_n), \text{ and}$$
  - if both the Pareto set and a point $p \in P$ are invariant under a permutation, the corresponding values c_i should be invariant too.
- Practitioners often use $F(v_1, \ldots) = \sum\limits_i v_i$, which means considering linear combinations of objective functions.
- The problem is that:
  - if the set S of possible values of $(f_1(x), \ldots, f_n(x))$ is not convex,
  - optimizing linear combinations does not cover the whole Pareto set (an example is given later).
- So what shall we do?

4. A reasonably simple solution: use min instead the sum

- One can easily show that if we use $F(v_1, \ldots) = \min(v_1, \ldots, v_n)$, then we do cover the whole Pareto set.
- Indeed, each point $x_0 \in P$ is covered when we take $c_i = 1/f_i(x_0)$.
- In this case, for $c_i = 1/f_i(x_0)$, for the function $g(x) \stackrel{\text{def}}{=} F(c_1 \cdot f_1(x), \ldots)$, we have $g(x_0) = 1$.
- However, for every other y, we will have $g(y) \le 1$ – otherwise y would dominate x_0.
- Of course, for any $c > 0$, the function $c \cdot \min(v_1, \ldots)$ has the same property.

5. A natural mathematical question: are $c \cdot \min$ the only functions with this property?

- Our answer is "yes". Here is a proof for $n = 2$.
- Let us first prove that $F(1, a) = F(1, 1)$ for all $a > 1$.
- Indeed, let us consider the Pareto set consisting of line segments

$$(0, a) - (1, a) - p = (1 + \varepsilon, 1 + \varepsilon) - (a, 1) - (a, 0).$$

- Both set P and the point p do not change when we swap v_1 and v_2.
- So we should have $c_1 = c_2$ for the values that lead to the maximum at p.
- Since we have a maximum at p, we get

$$F(c_1 \cdot (1 + \varepsilon), c_1 \cdot (1 + \varepsilon)) \geq F(c_1 \cdot 1, c_1 \cdot a).$$

- Hence due to homogeneity $F(1 + \varepsilon, 1 + \varepsilon) \geq F(1, a)$.
- In the limit $\varepsilon \to 0$, we get $F(1, 1) \geq F(1, a)$.

6. Are $c \cdot \min$ the only functions with this property?

- However, due to monotonicity, $F(1,1) \leq F(1,a)$, so $F(1,a) = F(1,1)$ for all $a > 1$.
- For any v, due to homogeneity, we have $F(v,v) = v \cdot F(1,1) = c \cdot v$.
- Similarly for any $v_1 < v_2$, we have $F(v_1, v_2) = v_1 \cdot F(1, v_2/v_1)$.
- Since $v_2 > v_1$, we have $v_2/v_1 > 1$, so $F(1, v_2/v_1) = F(1,1) = c$ and thus, $F(v_1, v_2) = c \cdot v_1$.
- Due to symmetry, for $v_1 > v_2$, we have $F(v_1, v_2) = F(v_2, v_1)$ and thus, $F(v_1, v_2) = c \cdot v_2$.
- In both cases, $F(v_1, v_2) = c \cdot \min(v_1, v_2)$.
- The statement is proven.

7. Acknowledgments

This work was supported in part by:

- National Science Foundation grants 1623190, HRD-1834620, HRD-2034030, and EAR-2225395;
- AT&T Fellowship in Information Technology;
- program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and
- a grant from the Hungarian National Research, Development and Innovation Office (NRDI).