Multi-Objective Optimization: Linear Combinations Do Not Cover Pareto Set, So What Does?

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1. Formulation of the problem

- In some practical situation, we have a clear objective: to optimize a known objective function $f(x)$ – e.g., profit for companies.
- Many effective algorithms are known for solving such well-defined optimization problems.
- Often, however, the problems are not so well-defined: there are several different objective functions $f_1(x), \ldots, f_n(x)$.
- Usually, we calibrate them so that the status quo state $s$ (when we do not make any decision) corresponds to $f_i(s) = 0$.
- This way, we are only looking for alternatives $x$ for which $f_i(x) \geq 0$ for all $i$.
- In such situations, we do not want alternatives $x$ which are dominated by others, i.e., for which, for some $y$:
  - we have $f_i(x) \leq f_i(y)$ for all $i$ and
  - we have $f_j(x) < f_j(y)$ for some $j$. 

2. Formulation of the problem (cont-d)

- We thus want to generate the set $P$ of all non-dominated alternatives, so that a human decision maker can make a final decision.
- This problem is known as *multi-objective optimization*.
- The set $P$ is known as the *Pareto set*.
- There are many effective algorithms for solving traditional optimization problem.
- So, a natural idea is to reduce multi-objective optimization to several optimization ones.
- Namely, we select a continuous function $F(v_1, \ldots, v_n)$ which:
  - is (non-strictly) increasing relative to each of its inputs and
  - which does not change under permutations.
- Then, we optimize functions $F(c_1 \cdot f_1(x), \ldots, c_n \cdot f_n(x))$ corresponding to different $c_i \geq 0$. 
3. Formulation of the problem (cont-d)

- It is also reasonable to require that:
  - the function $F$ is *homogeneous*, i.e., that for all $c$ and all $v_i$, we have:
    $$F(c \cdot v_1, \ldots, c \cdot v_n) = c \cdot F(v_1, \ldots, v_n),$$
  - if both the Pareto set and a point $p \in P$ are invariant under a permutation, the corresponding values $c_i$ should be invariant too.

- Practitioners often use $F(v_1, \ldots) = \sum_i v_i$, which means considering linear combinations of objective functions.

- The problem is that:
  - if the set $S$ of possible values of $(f_1(x), \ldots, f_n(x))$ is not convex,
  - optimizing linear combinations does not cover the whole Pareto set (an example is given later).

- So what shall we do?
4. A reasonably simple solution: use min instead the sum

- One can easily show that if we use \( F(v_1, \ldots) = \min(v_1, \ldots, v_n) \), then we do cover the whole Pareto set.

- Indeed, each point \( x_0 \in P \) is covered when we take \( c_i = 1/f_i(x_0) \).

- In this case, for \( c_i = 1/f_i(x_0) \), for the function \( g(x) \overset{\text{def}}{=} F(c_1 \cdot f_1(x), \ldots) \), we have \( g(x_0) = 1 \).

- However, for every other \( y \), we will have \( g(y) \leq 1 \) – otherwise \( y \) would dominate \( x_0 \).

- Of course, for any \( c > 0 \), the function \( c \cdot \min(v_1, \ldots) \) has the same property.
5. A natural mathematical question: are \( c \cdot \min \) the only functions with this property?

- Our answer is “yes”. Here is a proof for \( n = 2 \).
- Let us first prove that \( F(1, a) = F(1, 1) \) for all \( a > 1 \).
- Indeed, let us consider the Pareto set consisting of line segments

\[
(0, a) - (1, a) - p = (1 + \varepsilon, 1 + \varepsilon) - (a, 1) - (a, 0).
\]

- Both set \( P \) and the point \( p \) do not change when we swap \( v_1 \) and \( v_2 \).
- So we should have \( c_1 = c_2 \) for the values that lead to the maximum at \( p \).
- Since we have a maximum at \( p \), we get

\[
F(c_1 \cdot (1 + \varepsilon), c_1 \cdot (1 + \varepsilon)) \geq F(c_1 \cdot 1, c_1 \cdot a).
\]

- Hence due to homogeneity \( F(1 + \varepsilon, 1 + \varepsilon) \geq F(1, a) \).
- In the limit \( \varepsilon \to 0 \), we get \( F(1, 1) \geq F(1, a) \).
6. Are $c \cdot \min$ the only functions with this property?

- However, due to monotonicity, $F(1, 1) \leq F(1, a)$, so $F(1, a) = F(1, 1)$ for all $a > 1$.
- For any $v$, due to homogeneity, we have $F(v, v) = v \cdot F(1, 1) = c \cdot v$.
- Similarly for any $v_1 < v_2$, we have $F(v_1, v_2) = v_1 \cdot F(1, v_2/v_1)$.
- Since $v_2 > v_1$, we have $v_2/v_1 > 1$, so $F(1, v_2/v_1) = F(1, 1) = c$ and thus, $F(v_1, v_2) = c \cdot v_1$.
- Due to symmetry, for $v_1 > v_2$, we have $F(v_1, v_2) = F(v_2, v_1)$ and thus, $F(v_1, v_2) = c \cdot v_2$.
- In both cases, $F(v_1, v_2) = c \cdot \min(v_1, v_2)$.
- The statement is proven.
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