How Can There Be Objective Imprecise Probability

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1. Subjective Imprecise Probabilities: What Is Known

- Often, we only have partial information about the corresponding probabilities.
- This is known as *imprecise probabilities.*
- In general, the basic type of uncertainty is interval uncertainty.
- In line with this, the basic type of probabilistic uncertainty is interval-valued probabilities.
- One of the main ways to describe a probability distribution is by a cumulative distribution function (cdf) $F(x) \stackrel{\text{def}}{=} \text{Prob}(X \le x)$.
- A natural idea is thus to consider, for each x, an interval $[\underline{F}(x), \overline{F}(x)]$ of possible values of $F(x)$.
- This is known as *probability box*, or *p-box*, for short.
- p-boxes have been successfully used in many applications – as well as fuzzy-valued probabilities.

2. Objective Imprecise Probabilities: What Are They

- How can we have objective uncertainty in probability values?
- To understand this, let us recall what is probability from the practical viewpoint.
- In practice, probability p means, in effect, a frequency.
- We have a large number N of similar events (e.g., flipping a coin).
- These can be similar events occurring at different location and/or at different times.
- Probability p of a certain outcome means that this outcome is observed in $\approx p \cdot N$ cases.
- An ideal case is when the event settings are absolutely identical.

3. Objective Imprecise Probabilities (cont-d)

- For example:
  - we have a large set of identical atoms of a radioactive element, and
  - we observe how many of them emit radiation during a given period of time.
- In the usual quantum description, all the atoms are identical.
- However, the true quantum description is more complex.
- In quantum physics, the main idea is that everything is a matter of probability.

4. Objective Imprecise Probabilities (cont-d)

- In the first approximation – traditional quantum mechanics:
  - particle locations and velocities are only known with probabilities,
  - they can fluctuate around their classical values,
  - but the forces between particles are described by the usual formulas, e.g., Coulomb law

$$F = -c \cdot \frac{q_1 \cdot q_2}{r^2}.$$

- In secondary quantization, we take into account that the forces can also fluctuate around the classical values.
- In other words, the fields – that describe these forces – are also quantum objects whose values are only known with some probabilities.
- In general, no matter what kind of events we consider, these events are not identical.
- There are always quantum fluctuations because of which, for each event, the probability p_i is slightly different from p.

5. Objective Imprecise Probabilities (cont-d)

- Here, the values p_i are randomly fluctuating around the classical value p.
- In other words, here, we have objective imprecise probabilities.
- What does this mean in terms of observations?
- Can we experimentally detect the difference between precise and imprecise probabilities?
- To answer this question, let us recall what randomness means in terms of observations.

6. What Does Randomness Mean in Terms of Observations: Reminder

- What does randomness mean in terms of observations?
- Randomness means more than frequency.
- For example, according to Central Limit Theorem, differences between frequency and probability should be normally distributed.
- The general idea is that if a sequence is random, it must satisfy all the probability laws.
- A probability law is something that happens with probability 1.
- In mathematical terms, it is a set of probability measure 1 – so that its complement has measure 0.

7. What Does Randomness Mean in Terms of Observations (cont-d)

- So, a sequence is random if:
  - it does not belong to any definable set of probability measure 0,
  - or, equivalently, it does not belong to the union of all definable sets of measure 0.
- This is Kolmogorov's definition of a random sequence.
- Every definable set is described by a finite text – its definition.
- There are only countably many texts, so there are only countably many definable sets.
- The union of countably many sets of measure 0 still has measure 0.
- So, almost all sequence are random.

8. So Can We Experimentally Detect the Difference Between Precise and Imprecise Probabilities?

- We are interested in a sequence of events.
- Let $n_i = 1$ is the selected outcome occurred and $n_i = 0$ if it did not.
- We compare two cases:
  - precise case when each n_i occurs with probability p, and
  - imprecise case when each n_i occurs with probability p_i.
- Here:
  - we select some distribution on the set of all probabilities with mean p, and
  - take, as p_i, a random sequence of independent values corresponding to these probabilities.

9. Can We Experimentally Detect the Difference Between Type-1 and Imprecise Probabilities (cont-d)

- For both sequences, we can compare moments, i.e., averages over i from 1 to N of products $n_i^{a_1} \cdot n_{i+i_1}^{a_2} \cdot \ldots$.
- For example, mean is the average of n_i, covariance with next neighbor depends on $n_i \cdot n_{i+1}$, etc.
- Our *first result* is that for both sequences, each moment tends to the same limit: e.g., the mean tends to p.
- However, our *second result* is that for no sequence can be random with respect to both type-1 and imprecise distributions.
- This means that there are probability laws that are only true for "imprecise" sequences.
- So, it *is* possible to experimentally detect the difference!

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