How to Combine Expert Estimates

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1. Formulation of the problem

- In many practical situations, we rely on experts to estimate the probability of some future event E.
- To get a more accurate estimate, we ask several (n) experts and get their estimates $p_1, \ldots, p_n$.
- It is desirable to come up with a single estimate p that combines these estimates.

2. Reasonable assumption of independence

- Sometimes, the results of two experts are strongly correlated.
- Then, it makes not much sense to ask both.
- Indeed, the probability provided by the second expert will be practically the same as the probability of the first expert.
- Thus, it makes sense to assume that the experts are independent.

3. Analysis of the problem

- To analyze this situation, let us consider a more general situation, when:
  - we have n events $E_1, \ldots, E_n$, and
  - expert i estimates the probability of the i-th event E_i.
- In this case, we have 2^n possible situations:

$$E_1 \,\&\, \ldots \,\&\, E_{n-1} \,\&\, E_n,\ E_1 \,\&\, \ldots \,\&\, E_{n-1} \,\&\, \neg E_n,$$
$$E_1 \,\&\, \ldots \,\&\, E_{n-1} \,\&\, \neg E_{n-1} \,\&\, E_n,\ \ldots,\ \neg E_1 \,\&\, \ldots \,\&\, \neg E_{n-1} \,\&\, E_n.$$

- Since the experts are independent, the probability of each situation is equal to the product of the probabilities corresponding to each E_i:

$$p_1 \cdot \ldots \cdot p_{n-1} \cdot p_n,\ p_1 \cdot \ldots \cdot p_{n-1} \cdot (1 - p_n),\ p_1 \cdot \ldots \cdot p_{n-2} \cdot (1 - p_{n-1}) \cdot p_n,$$
$$\ldots,\ (1 - p_1) \cdot \ldots \cdot (1 - p_{n-1}) \cdot (1 - p_n).$$

4. Analysis of the problem and the resulting formula

- In our case, when there is only one event, we cannot have this event both happening and not happening.
- We have only two possible situations:
  - when E happens – with probability $p_1 \cdot \ldots \cdot p_{n-1} \cdot p_n$.
  - when E does not happen – with probability

$$(1-p_1) \cdot \ldots \cdot (1-p_{n-1}) \cdot (1-p_n).$$

- All other cases are inconsistent.
- Thus, under this condition of consistency, the (conditional) probability p that E will happen is equal to

$$p = \frac{p_1 \cdot \ldots \cdot p_{n-1} \cdot p_n}{p_1 \cdot \ldots \cdot p_{n-1} \cdot p_n + (1-p_1) \cdot \ldots \cdot (1-p_{n-1}) \cdot (1-p_n)}.$$

5. A natural question: when is the combined probability p larger than an individual estimate p_i – and is $p < p_i$?

- If we plug in the expression for p into the inequality $p > p_i$ and divide both sides by p_i, we get

$$\frac{p_1 \cdot \ldots \cdot p_{i-1} \cdot p_i \cdot \ldots \cdot p_{n-1} \cdot p_n}{p_1 \cdot \ldots \cdot p_{n-1} \cdot p_n + (1 - p_1) \cdot \ldots \cdot (1 - p_{n-1}) \cdot (1 - p_n)} > 1.$$

- If we then multiply both sides by the denominator, we get

$$p_1 \cdot \ldots \cdot p_{i-1} \cdot p_i \cdot \ldots \cdot p_{n-1} \cdot p_n > p_1 \cdot \ldots \cdot p_{n-1} \cdot p_n + (1 - p_1) \cdot \ldots \cdot (1 - p_{n-1}) \cdot (1 - p_n).$$

- The first term in the right-hand side can be obtained from the left-hand side by multiplying it by p_i.
- Thus, if we move this term to the left0-hand side and take this fact into account, we conclude that

$$(1 - p_i) \cdot p_1 \cdot \ldots \cdot p_{i-1} \cdot p_i \cdot \ldots \cdot p_{n-1} \cdot p_n > (1 - p_1) \cdot \ldots \cdot (1 - p_{n-1}) \cdot (1 - p_n).$$

6. A natural question (cont-d)

- Dididing both sides by $1-p_i$, we get the following equivalent inequality:

$$p_1\cdot\ldots\cdot p_{i-1}\cdot p_{i+1}\cdot\ldots p_n > (1-p_1)\cdot\ldots\cdot(1-p_{i-1})\cdot(1-p_{i+1})\cdot\ldots(1-p_n).$$

- This is equivalent to $o_1\cdot\ldots\cdot o_{i-1}\cdot o_{i+1}\cdot\ldots o_n > 1$, where the odds o_j are defined as $o_j \stackrel{\text{def}}{=} p_j/(1-p_j)$.
- In particular, for $n=2$ and $i=1$, this condition is equivalent to $p_2 > 1-p_2$, i.e., equivalently, to $p_2 > 0.5$.
- This makes perfect sense: $p_2 > 0.5$ means that the second expert is more confident that E will happen than that it will not happen.
- This positive belief in E increase the coverall probability of E.
- Vice versa, if the second expert is more negative about E, this decreases our confidence that E will happen.

7. A natural question (cont-d)

- For $n = 3$ and $i = 1$, the above inequality takes the form

$$p_2 \cdot p_3 > (1 - p_2) \cdot (1 - p_3).$$

- If we open the parentheses and subtract $p_2 \cdot p_3$ from both sides, we get $0 > 1 - p_2 - p_3$.
- This is equivalent to $p_2 + p_3 > 1$ or $(p_2 + p_3)/2 > 0.5$.
- In this case, our confidence increases if, on average, the other two experts believe in E more than in $\neg E$.

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