Why non-invertible symmetries

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1. Symmetries are important

- One of the main objectives of science and engineering is:
  - to predict the future (this is the science part) – and
  - to use this prediction ability to perform actions and/or design gadgets that will make the future better (this is the engineering part).
- The main idea behind prediction is that we expect the outcome to be the same as in similar situations in the past.
- Here, "similar" means that:
  - there may be some differences between the current situation and the past situations,
  - there may be some changes (= transformations),
  - but these changes do not affect the desired behavior.

2. Symmetries are important (cont-d)

- For example, in the past, the water boiled at 100°.
- We can move around, we can change the color of the teapot, we can rotate it – the result will be the same.
- Once the temperature reaches 100°, water starts boiling.
- In physics, such transformations that do not affect the analyzed phenomenon are known as *symmetries*.

3. Symmetries are usually invertible

- Most physical symmetries are invertible.
- If we move 100 m in one direction, we can always move back and thus get back to the original state.
- If we turn 90 degrees to the right, we can always turn back.
- We can replace all positive electric changes with negative ones and vice versa.
- This transformation preserves all electromagnetic interactions.
- Then we can replace them back and thus, get back to the original state.

4. Non-invertible symmetries emerge

- Recently, it was discovered that in quantum systems, an important role is played by non-invertible symmetries.
- There, two different states may be transformed into the same state.
- For example, we can consider an invertible symmetry S:
  - that swaps $|0\rangle$ and $|1\rangle$ states of a qubit
  - and that applies a similar swap to all the qubits in a multi-qubit system.
- It also makes sense to consider a transformation

$$T : x \mapsto \frac{1}{\sqrt{2}} \cdot x + \frac{1}{\sqrt{2}} \cdot S(x).$$

5. Non-invertible symmetries emerge

- This transformation also preserves some important properties of a physical system.
- For example, for the 2-qubit state $x = |01\rangle$, the resulting state is

$$T(x) = \frac{1}{\sqrt{2}} \cdot |01\rangle + \frac{1}{\sqrt{2}} \cdot |10\rangle.$$

- But we get the same result for $x = |10\rangle$.

6. Formulation of the problem: non-invertible symmetries help, but why?

- In several applications, non-invertible symmetries turnn out to be more helpful than invertible ones.
- A natural question is: why?

7. Our explanation

- Our explanation is based on calculus.
- According to calculus, the maximum of a function on a domain D is attained either in D's interior or on its border.
- If the maximum is attained in the interior, then:
  - at the corresponding point,
  - all partial derivatives of the maximized function should be equal to 0.
- In mathematical terms, this point should be a stationary point of this function.
- Many functions have only a few stationary points.
- So, if the domain is small, the probability that this domain contains one of the stationary points is also small.
- Thus, in most cases, the maximum is attained on the border of the domain.

8. Our explanation (cont-d)

- In our case, the domain D is the set all invertible transformations T that preserve certain properties.
- What function are we trying to maximize?
- Some characteristic of usefulness of the transformation T in analyzing the corresponding physical phenomena.
- We can therefore conclude that in many cases, the most effective transformations are the ones located on the border of D.
- Any invertible linear transformation is in the interior of D.
- Indeed, a small perturbation of an invertible matrix keeps it invertible.
- Thus, the border of D consists of non-invertible transformations.

9. Reference

- S.-H. Shao, "Noninvertible symmetries: what's done cannot be undone", *Physics Today*, July 2025, pp. 30–25.

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