An Ancient Bankruptcy Solution Makes Economic Sense

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1. The Bankruptcy Problem: Reminder

- When a person or a company cannot pay all its obligation:
  - a bankruptcy is declared, and
  - the available funds are distributed among the claimants.
- There is not enough money to give, to each claimant, what he/she is owed.
- So, claimants will get less than what they are owed.
- How much less?
- What is a fair way to divide the available funds between the claimants?

2. An Ancient Solution

- The bankruptcy problem is known for many millennia:
  - since money became available and
  - people starting lending money to each other.
- Solutions to this problem have also been proposed for many millennia.
- One such ancient solution is described in the Talmud, an ancient commentary on the Jewish Bible.
- This solution is described in the Babylonian Talmud, in Ketubot 93a, Bava Metzia 2a, and Yevamot 38a.
- This solution is actually about a more general problem of several contracts which cannot be all fully fulfilled.
- Like many ancient texts containing mathematics, the Talmud does not contain an explicit algorithm.

3. An Ancient Solution (cont-d)

- Instead, it contains four examples illustrating the main idea.
- In the first three examples, the three parties are owed the following amounts:
  - the first person is owed $d_1 = 100$ monetary units,
  - the second person is owed $d_2 = 200$ monetary units, and
  - the third person is owed $d_3 = 300$ monetary units:

$$d_1 = 100, \quad d_2 = 200, \quad d_3 = 300.$$

4. An Ancient Solution (cont-d)

- For three different available amounts E, the text describes the amounts e_1, e_2, and e_3 that each gets:

	$d_1 = 100$	$d_2 = 200$	$d_3 = 300$
E	e_1	e_2	e_3
100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
200	50	75	75
300	50	100	150

- There is also a fourth example, formulated in a slightly different way – as dividing a disputed garment.

5. An Ancient Solution (cont-d)

- In the bankruptcy terms, it can be described as follows: the owed amounts are: $d_1 = 50$, $d_2 = 100$.
- The available amount E and the recommended division (e_1, e_2) are as follows:

	$d_1 = 50$	$d_2 = 100$
E	e_1	e_2
100	25	75

6. Examples Are Here, But What is a General Solution?

- In many other ancient mathematical texts, where the general algorithm is very clear from the examples.
- However, in this particular case, the general algorithm was unknown until 1985.
- Actually, many researchers came up with algorithms that:
  - explained *some* of these examples,
  - while claiming that the original ancient text must have contained some mistakes.

7. Mystery Solved, Algorithm Is Reconstructed

- This problem intrigued Robert Aumann, later the Nobel Prize winner in Economics (2005).
- He came up with a reasonable general algorithm that explains the ancient solution.
- To explain this algorithm, we need to first start with the the case of two claimants.
- Without losing generality, let us assume that the first claimant has a smaller claim $d_1 \le d_2$.
- The first case is when the overall amount E is small – smaller that d_1.
- Then, the amount E is distributed equally between the claimants, so that each gets $e_1 = e_2 = \dfrac{E}{2}$.

8. Mystery Solved (cont-d)

- When the available amount E is between d_1 and d_2, i.e., when $d_1 \le E \le d_2$, then:
  - the first claimant receives $e_1 = \dfrac{d_1}{2}$, and
  - the second claimant receives the remaining amount $e_2 = E - e_1$.
- This policy continues until we reach the amount $E = d_2$, at which moment:
  - the first claimant receives the amount $d_1 = \dfrac{d_1}{2}$ and
  - the second claimant receives $e_2 = d_2 - \dfrac{d_1}{2}$.
- At this moment, after receiving the money, both claimants lose the same amount of money:

$$d_1 - e_1 = d_2 - e_2 = \frac{d_1}{2}.$$

9. Mystery Solved (cont-d)

- The third case is when E larger than d_2 (but smaller than the overall amount of debt $d_1 + d_2$).
- Then, the money is distributed in such a way that the losses remain equal, i.e., that

$$d_1 - e_1 = d_2 - e_2 \text{ and } e_1 + e_2 = E.$$

- From these two conditions, we get:

$$e_1 = \frac{E + d_1 - d_2}{2}, \quad e_2 = \frac{E - d_1 + d_2}{2}.$$

- The division between three (or more) claimants is then explained as the one for which:
  - for every two claimants,
  - the amounts given to them are distributed according to the above algorithm.

10. Mystery Solved (cont-d)

- This can be easily checked if we select,
  - for each pair (i, j)
  - only the overall amount $E_{ij} = e_i + e_j$ allocated to claimants from this pair.
- As a result, for the pairs $(1, 2)$, $(2, 3)$, and $(1, 3)$, we get the following tables:

	$d_1 = 100$	$d_2 = 200$
E_{12}	e_1	e_2
$66\frac{2}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
125	50	75
150	50	100

11. Mystery Solved (cont-d)

	$d_2 = 200$	$d_3 = 300$
E_{23}	e_2	e_3
$66\frac{2}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
150	75	75
250	100	150

	$d_1 = 100$	$d_3 = 300$
E_{13}	e_1	e_3
100	$66\frac{2}{3}$	$33\frac{1}{3}$
125	50	75
200	50	150

12. Remaining Problem

- The algorithm has been reconstructed, great.
- We now know *what* the ancients proposed.
- However, based on the above description, it is still not clear *why* this solution was proposed.
- The above solution sounds rather arbitrary.
- To be more precise:
  - both idea of dividing the amount equally and dividing the losses equally make sense, but
  - how do we combine these two ideas?

13. Remaining Problem (cont-d)

- And why in the region between $E = \min(d_1, d_2)$ and $E = \max(d_1, d_2)$,
  - the claimant with the smallest claim always gets half of his/her claim
  - while the second claimant gets more and more?
- How dow that fit with the Talmud's claim that the proposed division represents fairness?
- In this talk, we propose an economics-based explanation for the above solution.

14. Analysis of the Problem

- At first glance, it may look like fairness means dividing the amount either equally.
- If everyone is equal, why should someone gets more than others?
- However, this is not necessarily a fair division.
- Suppose that two folks start with an equal amount of 400 dollars.
- They both decided to invest some money in the biomedical company that:
  - promised to use this money
  - to develop a new drug curing up-to-now un-curable disease.
- The 1st person invested $200, the 2nd invested $300.

15. Analysis of the Problem (cont-d)

- After this, the first person has \$200 left and the second person has \$100 left.
- The company went bankrupt, and only \$300 remains in its account.
- If we divide this mount equally, both investors will get back the same amount of \$150.
- As a result:
  - the first person will have \$350 instead of the original \$400, while
  - the second person will have \$250 instead of the original \$400.

16. Analysis of the Problem (cont-d)

- So, the first person loses only \$50, while the second person loses three times more: \$150; so,
  - the first person, who selfishly kept money to himself, gets more than
  - the altruistic second person who invested more in a noble case;
  - how is this fair?

17. Let Us Divide Equally, But With Respect to What Status Quo Point?

- If two people jointly find an amount of money, then fairness means dividing equally.
- If two people jointly contributed to some expenses, fairness means that they should split the expenses equally.
- In both cases, we have a natural status quo point $(\widetilde{e}_1, \widetilde{e}_2)$:
  - in the first case, we take $(\widetilde{e}_1, \widetilde{e}_2) = (0, 0)$, and
  - in the second case, we take $(\widetilde{e}_1, \widetilde{e}_2) = (d_1, d_2)$.
- Any change from the status quo should be divided equally, i.e., we should have $e_1 - \widetilde{e}_1 = e_2 - \widetilde{e}_2$.
- This idea comes from another Nobelist, John Nash.
- So, to apply this idea to the bankruptcy problem, we need to decide what is the status quo point here.

18. Possible Ranges for Status Quo: Example

- Let us consider one of the above cases, when:
  - the first person is owed $d_1 = 100$ monetary units,
  - the second person is owed $d_2 = 200$ units, and
  - we have an amount $E_{12} = 125$ units to distribute between these two claimants.
- Depending on how we distribute this amount, the first person may get different amounts.
- The best possible case for the 1st claimant is when he get all the money he is owed, i.e., $\overline{e}_1 = 100$ units.
- The worst possible case for the 1st claimant is when:
  - all the money goes to the 2nd person, and
  - the 1st gets nothing: $\underline{e}_1 = 0$.
- Thus, the status quo point for the first person is somewhere in the interval $[\underline{e}_1, \overline{e}_1] = [0, 100]$.

19. Possible Ranges for Status Quo (cont-d)

- Similarly, the best possible case for the 2nd person is when the 2nd person gets all the money: $\overline{e}_2 = 125$.
- The worst possible case for the second person is when:
  - the first claimant gets everything he is owed – i.e., all 100 units, and
  - the second person gets the remaining amount of $\underline{e}_2 = 125 - 100 - 25$ units.
- Thus, the status quo point for the second person is somewhere in the interval $[\underline{e}_2, \overline{e}_2] = [25, 125]$.
- Let us perform the same analysis in the general case.

20. What Are Possible Ranges for the Status Quo Point: General Case

- Without losing generality, let us assume that the 1st person is the one who is owed less, i.e., that $d_1 \le d_2$.
- We will consider three different cases:
  - when the amount E_{12} does not exceed d_1: $E_{12} \le d_1$;
  - when E_{12} is between d_1 and d_2: $d_1 \le E_{12} \le E_2$,
  - and when E_{12} exceeds d_2: $d_2 \le E_{12} \le d_1 + d_2$.
- Let us consider these three cases one by one.

21. Case When the Overall Amount Does Not Exceed the Smallest Claim

- Let us first consider the case when $E_{12} \le d_1 \le d_2$.
- In this case, for the first person, the best possible case is when this person gets all the amount E_{12}: $\overline{e}_1 = E_{12}$.
- The worst possible case is when all the money goes to the 2nd claimant and the 1st gets nothing: $\underline{e}_1 = 0$.
- So, for the first person, the range of possible gains is $[\underline{e}_1, \overline{e}_1] = [0, E_{12}]$.
- For the second person, the best possible case is when this person gets all the amount E_{12}: $\overline{e}_2 = E_{12}$.
- The worst possible case is when all the money goes to the 1st claimant and the 2nd gets nothing: $\underline{e}_2 = 0$.
- So, for the second person, the range of possible gains is $[\underline{e}_2, \overline{e}_2] = [0, E_{12}]$.

22. Case When the Overall Amount Is in Between the Smaller and the Larger Claims

- Let us now consider the case when $d_1 \le E_{12} \le d_2$.
- In this case, for the 1st person, the best case is when he/she gets all the amount owed: $\overline{e}_1 = d_1$.
- The worst case is when all the money goes to the 2nd claimant and the 1st gets nothing: $\underline{e}_1 = 0$.
- So, for the first person, the range of possible gains is

$$[\underline{e}_1, \overline{e}_1] = [0, d_1].$$

23. Case When the Overall Amount Is in Between the Smaller and the Larger Claims (cont-d)

- For the second person, the best possible case is when this person gets all the amount E_{12}: $\overline{e}_2 = E_{12}$.
- The worst possible case is when:
  - the first claimant gets all the money he is owed (i.e., the amount d_1), and
  - the second person only gets the remaining amount

$$\underline{e}_2 = E_{12} - d_1.$$

- So, for the second person, the range of possible gains is $[\underline{e}_2, \overline{e}_2] = [E_{12} - d_1, E_{12}]$.

24. Case When the Overall Amount Is Larger Than Both Claims

- Let us now consider the case when $d_1 \le d_2 \le E_{12}$.
- In this case, for the 1st person, the best case is when this person gets all the amount owed: $\overline{e}_1 = d_1$.
- The worst possible case is when:
  - the second person gets all the money it is owed, and
  - the first person only gets the remaining amount

$$\underline{e}_1 = E_{12} - d_2.$$

- So, for the first person, the range of possible gains is

$$[\underline{e}_1, \overline{e}_1] = [E_{12} - d_2, d_1].$$

25. Case When the Overall Amount Is Larger Than Both Claims (cont-d)

- For the second person, the best possible case is when this person gets all the amount it is owed: $\overline{e}_2 = d_2$.
- The worst possible case is when:
  - the first claimant gets all the money he is owed (i.e., the amount d_1), and
  - the second person only gets the remaining amount

$$\underline{e}_2 = E_{12} - d_1.$$

- So, for the second person, the range of possible gains is $[\underline{e}_2, \overline{e}_2] = [E_{12} - d_1, d_2]$.

26. Which Points of the Corresponding Intervals Should We Select?

- In all three cases, for both claimants, we have an *interval* of possible values of the resulting gain.
- On each of these intervals:
  - we need to select a status quo point
  - that corresponds to the equivalent cost of this interval uncertainty.
- This is a particular case of the problem of what is the fair cost $\overline{e}$ in the case of interval uncertainty $[\underline{e}, \overline{e}]$.
- This problem has been handled by yet another Nobelist, Leo Hurwicz.
- Namely, he proposed to select the value

$$\widetilde{e} = \alpha \cdot \overline{e} + (1 - \alpha) \cdot \underline{e}.$$

27. Which Points of the Corresponding Intervals Should We Select (cont-d)

- Here, $\alpha \in [0,1]$ describes the decision-maker's degree of optimism-pessimism.
- The value $\alpha = 1$ describes a perfect optimist, who only takes into account the best possible scenario.
- The value $\alpha = 0$ describes a complete pessimist, who only takes into account the worst possible scenario.
- Values α strictly between 0 and 1 describe a realistic decision maker.
- Let us see what will happen if:
  - we take one of these solutions as a status-quo point
  - and consider a division fair if the differences between the gains e_i and the status quo are equal:

$$e_1 - \widetilde{e}_1 = e_2 - \widetilde{e}_2.$$

28. No Matter What Our Level of Optimism, We Get Exactly the Ancient Solution

- We will now show that in all the cases, we get exactly the ancient solution.
- So, we have a good economic explanation for this solution.
- To show this, let us consider all three possible cases:
  - case when $E_{12} \le d_1 \le d_2$,
  - case when $d_1 \le E_{12} \le d_2$, and
  - case when $d_1 \le d_2 \le E_{12}$.

29. Case When the Overall Amount Does Not Exceed the Smallest Claim: General Formulas

- In this case,

  $$\widetilde{e}_1 = \alpha \cdot \overline{e}_1 + (1-\alpha) \cdot \underline{e}_1 = \alpha \cdot E_{12} + (1-\alpha) \cdot 0 = \alpha \cdot E_{12}$$

  $$\widetilde{e}_2 = \alpha \cdot \overline{e}_2 + (1-\alpha) \cdot \underline{e}_2 = \alpha \cdot E_{12} + (1-\alpha) \cdot 0 = \alpha \cdot E_{12}.$$

- Thus, the fairness condition $e_1 - \widetilde{e}_1 = e_2 - \widetilde{e}_2$ takes the form $e_1 - \alpha \cdot E_{12} = e_2 - \alpha \cdot E_{12}$, i.e., the form $e_1 = e_2$.
- So, in this case:
  - no matter what is the optimism-pessimism value α,
  - we divide the available amount E_{12} equally between the claimants: $e_1 = e_2 = \dfrac{E_{12}}{2}$.
- This is exactly what the ancient solution recommends in this case.

30. Case When the Overall Amount Does Not Exceed the Smallest Claim: Example

- Let us consider one of the above examples, when $d_1 = 100$, $d_2 = 200$, and $E_{12} = 66\frac{2}{3}$.
- In this case, the above formulas recommend a solution in which $e_1 = e_2 = 33\frac{1}{3}$.
- For the optimistic case $\alpha = 1$, the status quo point is

$$\widetilde{e}_1 = \overline{e}_1 = 66\frac{2}{3} \text{ and } \widetilde{e}_2 = \overline{e}_1 = 66\frac{2}{3}.$$

- Thus, the condition of fairness with respect to this optimistic status quo point is indeed satisfied:

$$e_1 - \widetilde{e}_1 = e_2 - \widetilde{e}_2 = -33\frac{1}{3}.$$

31. Case When $d_1 \le E_{12} \le d_2$: General Formulas

- In this case:

$$\widetilde{e}_1 = \alpha \cdot \overline{e}_1 + (1-\alpha) \cdot \underline{e}_1 = \alpha \cdot d_1 + (1-\alpha) \cdot 0 = \alpha \cdot d_1;$$

$$\widetilde{e}_2 = \alpha \cdot \overline{e}_2 + (1-\alpha) \cdot \underline{e}_2 = \alpha \cdot E_{12} + (1-\alpha) \cdot (E_{12} - d_1) = E_{12} - (1-\alpha) \cdot d_1.$$

- So, the fairness condition $e_1 - \widetilde{e}_1 = e_2 - \widetilde{e}_2$ becomes:

$$e_1 - \alpha \cdot d_1 = e_2 - E_{12} + (1-\alpha) \cdot d_1 = e_2 - E_{12} + d_1 - \alpha \cdot d_1.$$

- Canceling the common term $-\alpha \cdot d_1$ on both sides, we get $e_1 = e_2 - E_{12} + d_1$.
- Substituting $e_2 = E - e_1$ into this formula, we conclude that $e_1 = E_{12} - e_1 - E_{12} + d_1$, i.e., $e_1 = -e_1 + d_1$.
- Moving the term $-e_1$ to the left-hand side, we get $2e_1 = d_1$ and $e_1 = \dfrac{d_1}{2}$.
- The 2nd person gets the remaining amount $e_2 = E_{12} - \dfrac{d_1}{2}$ – exactly what the ancient solution recommends.

32. Case When $d_1 \le E_{12} \le d_2$: Example

- Let us consider one of the above examples, when $d_1 = 100$, $d_2 = 200$, and $E_{12} = 125$.
- In this case, the above formulas recommend a solution in which

$$e_1 = \frac{100}{2} = 50 \text{ and } e_2 = E_{12} - e_1 = 125 - 50 = 75.$$

- Here, the optimistic status quo point is $\widetilde{e}_1 = d_1 = 100$ and $\widetilde{e}_2 = E_{12} = 125$.
- Thus, the condition of fairness with respect to this optimistic status quo point is indeed satisfied:

$$e_1 - \widetilde{e}_1 = 50 - 100 = -50, \quad e_2 - \widetilde{e}_2 = 75 - 125 = -50.$$

33. Case When the Overall Amount Is Larger Than Both Claims: General Formulas

- In this case,

$$\widetilde{e}_1 = \alpha \cdot \overline{e}_1 + (1-\alpha) \cdot \underline{e}_1 = \alpha \cdot d_1 + (1-\alpha) \cdot (E_{12} - d_2) =$$

$$\alpha \cdot d_1 + (1-\alpha) \cdot E_{12} - (1-\alpha) \cdot d_2;$$

$$\widetilde{e}_2 = \alpha \cdot \overline{e}_2 + (1-\alpha) \cdot \underline{e}_2 = \alpha \cdot d_2 + (1-\alpha) \cdot (E_{12} - d_1) =$$

$$\alpha \cdot d_2 + (1-\alpha) \cdot E_{12} - (1-\alpha) \cdot d_1.$$

- So, the fairness condition $e_1 - \widetilde{e}_1 = e_2 - \widetilde{e}_2$ becomes:

$$e_1 - \alpha \cdot d_1 - (1-\alpha) \cdot E_{12} + (1-\alpha) \cdot d_2 =$$

$$e_2 - \alpha \cdot d_2 - (1-\alpha) \cdot E_{12} + (1-\alpha) \cdot d_1.$$

- Canceling the common term $-(1-\alpha) \cdot E_{12}$, we get

$$e_1 - \alpha \cdot d_1 + (1-\alpha) \cdot d_2 = e_2 - \alpha \cdot d_2 + (1-\alpha) \cdot d_1.$$

34. Case When $d_2 < E_{12}$ (cont-d)

- Moving terms containing d_1 and d_2 to the right-hand side, we conclude that $e_1 = e_2 + d_1 - d_2$.
- Substituting $e_2 = E_{12} - e_1$ into this formula, we get $e_1 = E_{12} - e_1 + d_1 - e_2$.
- Moving the term $-e_1$ to the left-hand side, we get $2e_1 = E_{12} + d_1 - e_2$ and $e_1 = \dfrac{E_{12} + d_1 - d_2}{2}$.
- The second person gets the remaining amount

$$e_2 = E_{12} - \frac{E_{12} + d_1 - d_2}{2} = \frac{E_{12} - d_1 + d_2}{2}.$$

- This too is exactly what the ancient solution recommends in this case.

35. Case When the Overall Amount Is Larger Than Both Claims: Example

- Let us consider one of the above examples, when

$$d_1 = 50, \quad d_2 = 100, \text{ and } E_{12} = 100.$$

- In this case, the above formulas recommend a solution in which

$$e_1 = \frac{100 + 50 - 100}{2} = 25 \text{ and } e_2 = \frac{100 - 50 + 100}{2} = 75.$$

- Here, the optimistic status quo point is $\widetilde{e}_1 = d_1 = 50$ and $\widetilde{e}_2 = d_2 = 100$.
- Thus, the condition of fairness with respect to this optimistic status quo point is indeed satisfied:

$$e_1 - \widetilde{e}_1 = 25 - 50 = -25, \quad e_2 - \widetilde{e}_2 = 75 - 100 = -25.$$

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