Decision Making Under Interval Uncertainty: Beyond Hurwicz Pessimism-Optimism Criterion

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1. Decision Making under Interval Uncertainty

- In the ideal case, we know the exact consequence of each action.
- In this case, a natural idea is to select an action that will lead to the largest profit.
- In real life, we rarely know the exact consequence of each action.
- In many cases, all we know are the lower and upper bound on the quantities describing such consequences.
- So, all we know is an interval $[\underline{u}, \overline{u}]$ that contains the actual (unknown) value u.
- So, we have several alternatives a for each of which we only have an interval estimate $[\underline{u}(a), \overline{u}(a)]$.
- Which alternative should we select?

2. Hurwicz's Idea

- The problem of decision making under interval uncertainty was first handled by a Nobelist Leo Hurwicz.
- Hurwicz's main idea was as follows.
- We know how to make decisions when for each alternative, we know the exact value of the resulting profit.
- So, to help decision makers make decisions under interval uncertainty, Hurwicz proposed:
  - to assign, to each interval $\mathbf{a} = [\underline{a}, \overline{a}]$, an equivalent value $u_H(\mathbf{a})$, and
  - then to select an alternative with the largest equivalent value.

3. Natural Requirements on u_H(a)

- Of course, when we know the exact consequence a, we should have $u_H([a, a]) = a$.
- All the values a from the interval $[\underline{a}, \overline{a}]$ are larger than (thus better than) or equal to the lower endpoint $\underline{a}$.
- So, the equivalent value must also be larger than or equal to $\underline{a}$.
- Similarly, all the values a from the interval $[\underline{a}, \overline{a}]$ are worse than or equal to the upper endpoint $\overline{a}$.
- So, the equivalent value must also be smaller than or equal to $\overline{a}$: $\underline{a} \le u_H([\underline{a}, \overline{a}]) \le \overline{a}$.

4. Scale Invariance

- The equivalent value should not change if we change a monetary unit.
- What was better when we count in dollars should also be better when we use Vietnamese Dongs instead.
- We can change from the original monetary unit to a new unit which is k times smaller.
- Then, all the numerical values are multiplied by k.
- Thus, if we have $u_H(\underline{a}, \overline{a}) = a_0$, then, for all $k > 0$, we should have $u_H([k \cdot \underline{a}, k \cdot \overline{a}]) = k \cdot a_0$.

5. Additivity

- Suppose that we have two separate independent situations, with possible profits $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$s.
- Then, the overall profit of these two situations can take any value from $\underline{a} + \underline{b}$ to $\overline{a} + \overline{b}$.
- So, we can compute the equivalent value of the corresponding interval

$$\mathbf{a} + \mathbf{b} \stackrel{\text{def}}{=} [\underline{a} + \underline{b}, \overline{a} + \overline{b}].$$

- Second, we can first find equivalent values of each of the intervals and then add them up.
- It is reasonable to require that the resulting value should be the same in both cases, i.e., that we should have

$$u_H([\underline{a} + \underline{b}, \overline{a} + \overline{b}]) = u_H([\underline{a}, \overline{a}]) + h_H([\underline{b}, \overline{b}]).$$

- This property is known as *additivity*.

6. Derivation of Hurwicz Formula

- Let us denote $\alpha_H \stackrel{\text{def}}{=} u_H([0,1])$; due to the first natural requirement, $0 \le \alpha_H \le 1$.
- Now, due to scale-invariance, for every value $a > 0$, we have $u_H([0,a]) = \alpha_H \cdot a$.
- For $a = 0$, this is also true, since in this case, we have

$$u_H([0,0]) = 0.$$

- In particular, for every two values $\underline{a} \le \overline{a}$, we have

$$u_H([0, \overline{a} - \underline{a}]) = \alpha_H \cdot (\overline{a} - \underline{a}).$$

- Now, we also have $u_H([\underline{a}, \underline{a}]) = \underline{a}$.
- Thus, by additivity, we get $u_H([\underline{a}, \overline{a}]) = (\overline{a} - \underline{a}) \cdot \alpha_H + \underline{a}$, i.e., $u_H([\underline{a}, \overline{a}]) = \alpha_H \cdot \overline{a} + (1 - \alpha_H) \cdot \underline{a}$.
- This is the formula for which Leo Hurwicz got his Nobel prize.

7. Meaning of Hurwicz Formula

- When $\alpha_H = 1$, this means that the equivalent value is equal to the largest possible value $\overline{a}$.
- So, when making a decision, the person only takes into account the best possible scenario.
- In real life, such a person is known as an *optimist*.
- When $\alpha_H = 0$, this means that the equivalent value is equal to the smallest possible value $\underline{a}$.
- So, when making a decision, the person only takes into account the worst possible scenario.
- In real life, such a person is known as an *pessimist*.
- When $0 < \alpha_H < 1$, this means that a person takes into account both good and bad possibilities.
- So, α_H is called *optimism-pessimism coefficient*, and the procedure *optimism-pessimism criterion.*

8. Need to Go Beyond Hurwicz Criterion

- While Hurwicz criterion is reasonable, it leaves several options equivalent which should not be equivalent.
- For example, if $\alpha_H = 0.5$, then, according to Hurwicz criterion, the interval $[-1, 1]$ should be equivalent to 0.
- However, a risk-averse person will prefer status quo (0) to a situation $[-1, 1]$ in which s/he can lose.
- A risk-prone person would prefer $[-1, 1]$ in which he/she can gain.
- To take this into account, we need to go beyond assigning a numerical value to each interval.
- We need, instead, to describe possible orders on the class of all intervals.
- This is what we do in this talk.

9. Analysis of the Problem

- For every two alternatives a and b, we want to provide one of the following three recommendations:
  - select the first alternative; we will denote this recommendation by $b < a$;
  - select the second alternative; we will denote this recommendation by $a < b$; or
  - treat these two alternatives as equivalent ones; we will denote this recommendation by $a \sim b$.
- Our recommendations should be consistent: e.g.,
  - if we recommend that b is preferable to a and that c is preferable to b,
  - then we should also recommend that c is preferable to a.
- Such consistency can be described by the following definition.

10. Analysis of the Problem (cont-d)

- *For every set* A, *by a* linear pre-order, *we mean a pair of relations* $(<, \sim)$ *such that:*
  - *for every* a *and* b, *exactly one of the three possibilities must be satisfied:* $a < b$, *or* $b < a$, *or* $a \sim b$;
  - *for all* a, *we have* $a \sim a$;
  - *for all* a *and* b, *if* $a \sim b$, *then* $b \sim a$;
  - *for all* a, b, *and* c, *if* $a \sim b$ *and* $b \sim c$, *then* $a \sim c$;
  - *for all* a, b, *and* c, *if* $a < b$ *and* $b < c$, *then* $a < c$;
  - *for all* a, b, *and* c, *if* $a < b$ *and* $b \sim c$, *then* $a \sim c$; *and*
  - *for all* a, b, *and* c, *if* $a \sim b$ *and* $b < c$, *then* $a < c$.
- To describe a linear pre-order, it is sufficient to describe when $a < b$: indeed, $a \sim b \Leftrightarrow (a \not< b \,\&\, b \not< a)$.
- We want to describe all possible linear pre-orders on the set of all possible intervals.

11. Analysis of the Problem (cont-d)

- When intervals are degenerate (i.e., are real numbers), this pre-order must coincide with the usual order.
- Also, similarly to the Hurwicz case, an interval $[\underline{a}, \overline{a}]$ cannot be worse than $\underline{a}$ and cannot be better than $\overline{a}$.
- Thus, we arrive at the following definition.
- *A linear pre-order on the set of all possible intervals* $\mathbf{a} = [\underline{a}, \overline{a}]$ *is called* natural *if:*
  - $\forall a\,\forall b\,([a, a] < [b, b] \Leftrightarrow a < b)$, *and*
  - $\forall \underline{a} \le \overline{a}\,([\underline{a}, \overline{a}] \not< [\underline{a}, \underline{a}]\,\&\,[\overline{a}, \overline{a}] \not< [\underline{a}, \overline{a}])$.

12. Scale Invariance and Additivity

- It is reasonable to require that our linear pre-order does not change if we change a monetary unit.
- *A linear pre-order on the set of all possible intervals is called* scale-invariant *if for all $k > 0$:*
  - *if* $[\underline{a}, \overline{a}] < [\underline{b}, \overline{b}]$, *then* $[k \cdot \underline{a}, k \cdot \overline{a}] < [k \cdot \underline{b}, k \cdot \overline{b}]$;
  - *if* $[\underline{a}, \overline{a}] \sim [\underline{b}, \overline{b}]$, *then* $[k \cdot \underline{a}, k \cdot \overline{a}] \sim [k \cdot \underline{b}, k \cdot \overline{b}]$.
- *A linear pre-order on the set of all possible intervals is called* additive *if for every three intervals* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$*:*
  - *if* $\mathbf{a} < \mathbf{b}$, *then* $\mathbf{a} + \mathbf{c} < \mathbf{b} + \mathbf{c}$;
  - *if* $\mathbf{a} \sim \mathbf{b}$, *then* $\mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$.

13. Main Result

- *Let* $(< . \sim)$ *be a natural scale-invariant additive linear pre-order on the set of all possible intervals.*
- *Then, there exists a number* α_H *for which the pre-order has one of the following three forms:*
  - $[\underline{a}, \overline{a}] < [\underline{b}, \overline{b}]$ *if and only if*

    $$\alpha_H \cdot \overline{a} + (1 - \alpha_H) \cdot \underline{a} < \alpha_H \cdot \overline{b} + (1 - \alpha_H) \cdot \underline{b};$$

  - $[\underline{a}, \overline{a}] < [\underline{b}, \overline{b}] \Leftrightarrow$ *either we have the above inequality, or we have an equality and* **a** *is wider than* **b**, *i.e.,*

    $$\overline{a} - \underline{a} > \overline{b} - \underline{b};$$

  - $[\underline{a}, \overline{a}] < [\underline{b}, \overline{b}] \Leftrightarrow$ *either we have the above inequality, or we have an equality and* **a** *is narrower than* **b**.
- *Vice versa, for each* $\alpha_H \in [0, 1]$, *all three relations are natural scale-invariant consistent pre-orders.*

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14. Discussion

- The first relation describes a risk-neutral decision maker.
- Then, all intervals with the same Hurwicz equivalent value are indeed equivalent.
- The second relation describes a risk-averse decision maker.
- The narrowest interval means that the risk is the smallest.
- Finally, the third relation describes a risk-prone decision maker.
- The widest interval means that the risk is the largest.

15. Relation to Non-Standard Analysis

- In *non-standard analysis*:
  - in addition to usual ("standard") real numbers,
  - we also have *infinitesimal* real numbers.
- E.g., we have objects ε which are positive but which are smaller than all positive standard real numbers.
- We can perform usual arithmetic operations on all the numbers, standard and others ("non-standard").
- In particular:
  - for every real number x and a positive infinitesimal number $\varepsilon > 0$,
  - we can consider non-standard numbers $x + \varepsilon$ and $x - \varepsilon$.

16. Non-Standard Analysis (cont-d)

- Vice versa:
  - if a non-standard real number is bounded from below and from above by standard real numbers,
  - then it can be represented in one of these two forms.
- Then, $x + k \cdot \varepsilon < x' + k' \cdot \varepsilon$ means that:
  - either $x < x'$
  - or $x = x'$ and $k < k'$.
- Thus, all three relations can be described in Hurwicz terms for some $\alpha_{NS} \in [0, 1]$

$$\mathbf{a} = [\underline{a}, \overline{a}] < \mathbf{b} = [\underline{b}, \overline{b}] \Leftrightarrow$$

$$(\alpha_{NS} \cdot \overline{a} + (1 - \alpha_{NS}) \cdot \underline{a} < \alpha_{NS} \cdot \overline{b} + (1 - \alpha_{NS}) \cdot \underline{b}).$$

- The only difference from the traditional Hurwicz approach is that now the value α_{NS} can be non-standard.

17. Non-Standard Analysis (cont-d)

- When α_{NS} is a standard real number, we get the usual Hurwicz ordering.
- When $\alpha_{NS} = \alpha_H - \varepsilon$, we get the risk-averse case.
- When $\alpha_{NS} = \alpha_H + \varepsilon$ for some standard real number α_H, we get the risk-prone case.

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19. Proof: First Case

- Let us start with the same interval $[0, 1]$ as in the above derivation of the Hurwicz criterion:
  - if the interval $[0, 1]$ is equivalent to some real number α_H – i.e., to the degenerate interval
$$[0, 1] \sim [\alpha_H, \alpha_H],$$
  - then we can conclude that every interval $[\underline{a}, \overline{a}]$ is equivalent to its Hurwicz equivalent value
$$u_H(\mathbf{a}) \stackrel{\text{def}}{=} \alpha_H \cdot \overline{a} + (1 - \alpha_H) \cdot \underline{a}.$$
- The proof is similar to the usual derivation of Hurwicz's formula.
- Here, because of naturalness, we have $\alpha_H \in [0, 1]$.
- This is the first option from our main result.

20. Proof: Remaining Cases

- Let us consider the cases when the interval $[0,1]$ is *not* equivalent to any real number.
- Since we consider a linear pre-order, this means that for every real number a, $a < [0,1]$ or $[0,a] < a$.
- If for some real number a, we have $a < [0,1]$, then $a' < [0,1]$ for all $a' < a$.
- This follows from transitivity and naturalness.
- Similarly, if for some real number b, we have $[0,1] < b$, then we have $[0,1] < b'$ for all $b' > b$.
- Thus, there is a threshold value $\alpha_H = \sup\{a : a < [0,1]\} = \inf\{b : [0,1] < b\}$ such that:
  - for $a < \alpha_H$, we have $a < [0,1]$, and
  - for $a > \alpha_H$, we have $[0,1] < a$.

21. Remaining Cases (cont-d)

- Because of naturalness, we have $\alpha_H \in [0, 1]$.
- We consider the case when the interval $[0, 1]$ is not equivalent to any real number.
- So, either $[0, 1] < \alpha_H$ or $\alpha_H < [0, 1]$.
- Let us consider these cases one by one.

22. Case When $[0,1] < \alpha_H$

- In this case, due to scale-invariance and additivity with $\mathbf{c} = [\underline{a}, \underline{a}]$, for every interval $[\underline{a}, \overline{a}]$, we have:
  - when $a < \alpha_H \cdot \overline{a} + (1 - \alpha_H) \cdot \underline{a}$, then $a < [\underline{a}, \overline{a}]$; and
  - when $a \geq \alpha_H \cdot \overline{a} + (1 - \alpha_H) \cdot \underline{a}$, then $[\underline{a}, \overline{a}] \leq a$.
- Thus, if $u_H(\mathbf{a}) < u_H(\mathbf{b})$, then

$$\mathbf{a} < \frac{u_H(\mathbf{a}) + u_H(\mathbf{b})}{2} < \mathbf{b}, \text{ hence } \mathbf{a} < \mathbf{b}.$$

- What if intervals have the same Hurwicz equivalent value?
- For every $k > 0$, the Hurwicz equivalent value of the interval $[-k \cdot \alpha_H, k \cdot (1 - \alpha_H)]$ is 0.
- Thus, we have $[-k \cdot \alpha_H, k \cdot (1 - \alpha_H)] < 0$.

23. Case When $[0,1] < \alpha_H$ (cont-d)

- We have $[-k\cdot\alpha_H, k\cdot(1-\alpha_H)] < 0$.
- So, for every $k' > 0$, by using additivity with $\mathbf{c} = [-k'\cdot\alpha_H, k'\cdot(1-\alpha_H)]$, we conclude that

$$[-(k+k')\cdot\alpha_H, (k+k')\cdot(1-\alpha_H)] < [-k\cdot\alpha_H, k\cdot(1-\alpha_H)].$$

- Hence, for two intervals with the same Hurwicz equivalent value 0, the narrower one is better.
- By applying additivity with $\mathbf{c} = u_H(\mathbf{a})$, we conclude that the same is true for all possible $u_H(\mathbf{a})$.
- This is the second case of our main result.

24. Case When $\alpha_H < [0, 1]$

- Similarly to the previous case, we can conclude that if $u_H(\mathbf{a}) < u_H(\mathbf{b})$, then $\mathbf{a} < \mathbf{b}$.
- Then, similarly to the previous case, we prove that when $u_H(\mathbf{a}) = u_H(\mathbf{b})$, the wider interval is better.
- This is the third option from our main result.