# Chubanov's Method – A New Polynomial-Time Algorithm for Linear Programming

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(based on a joint work with Olga Kosheleva)

# 1. Problems That We Solve in Real Life

- In many practical situations, we need to maximize or minimize some objective function.
- When we select a plan for a company, we want to maximize profit.
- When we select a route for a car, we want to minimize travel time.
- When we select medical treatment, we want to minimize side effects, etc.
- In all these situations, there are some constraints.
- Pollution generated by a chemical plant cannot exceed the legal limits.
- A car cannot exceed the speed limit – unless it is an emergency vehicle.

## 2. Real-Life Problems (cont-d)

- A medical treatment must satisfy a certain rate of cure, etc.
- In general, there are several parameters $x_1, \ldots, x_n$ possible alternatives.
- The objective function $f(x_1, \ldots, x_n)$ depends on all these parameters.
- A constraints means that some quantity $g$ cannot exceed the corresponding threshold $t$.
- This quantity also depends on the parameters $x_1, \ldots, x_n$: $g = g(x_1, \ldots, x_n)$.
- Thus, a constraint has the form $g(x_1, \ldots, x_n) \le t$.
- In general, we have a *constraint optimization* problem: maximize $f(x_1, \ldots, x_n)$ under constraints

$$g_1(x_1, \ldots, x_n) \le t_1, \ldots, g_m(x_1, \ldots, x_n) \le t_m.$$

## 3. Linearization Is Often Possible

- In many practical situations, we know a reasonable good solution $x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})$.
- This usually means that the unknown optimal solution $x = (x_1, \ldots, x_n)$ is close to $x^{(0)}$.
- In other words, the differences $v_i \stackrel{\text{def}}{=} x_i - x_i^{(0)}$ are small.
- In physics and engineering, if the quantities $v_i$ are small, we can safely ignore terms quadratic in $v_i$.
- For example, if $v_i \approx 10\%$, then $v_i^2 \approx 1\% \ll 10\%$.

# 4. Linearization (cont-d)

- Thus, we can, e.g.:
  - take the expression

$$f(x_1, \ldots, x_n) = f\left(x_1^{(0)} + v_1, \ldots, x_n^{(0)} + v_n\right);$$

  - expand it in Taylor series and keep only linear terms in this expansion:

$$f(x_1, \ldots, x_n) \approx y^{(0)} + \sum_{j=1}^{n} c_i \cdot v_i,$$

where $y^{(0)} \stackrel{\text{def}}{=} f\left(x_1^{(0)}, \ldots, x_n^{(0)}\right)$ and $c_j \stackrel{\text{def}}{=} \dfrac{\partial f}{\partial x_j}$.

- Maximizing this expression for $f(x_1, \ldots, x_n)$ is equivalent to maximizing a linear function $\sum\limits_{j=1}^{n} c_i \cdot v_i$.

## 5. Linearization (cont-d)

- By applying a similar linearization to $g_i(x_1, \ldots, x_n) = g_i\left(x_1^{(0)} + v_1, \ldots, x_n^{(0)} + v_n\right)$, we conclude that

$$g_i(x_1, \ldots, x_n) \approx g_{i0} + \sum_{j=1}^{m} a_{ij} \cdot v_j,$$

$$\text{where } g_{i0} \stackrel{\text{def}}{=} g_i\left(x_1^{(0)}, \ldots, x_n^{(0)}\right) \text{ and } a_{ij} \stackrel{\text{def}}{=} \frac{\partial g_i}{\partial x_j}.$$

- Thus, each constraint $g_i(x_1, \ldots, x_n) \le t_i$ takes the form $\sum\limits_{j=1}^{n} a_{ij} \cdot v_j \le b_i$, where $b_i \stackrel{\text{def}}{=} t_i - g_{i0}$.
- Thus, we arrive at the problem of maximizing a linear function $\sum\limits_{j=1}^{n} c_i \cdot v_i$ under linear constraints $\sum\limits_{j=1}^{n} a_{ij} \cdot v_j \le b_i$.
- Such problems are known as *linear programming*.

# 6. Why the Name?

- Why linear – clear, but why programming?
- The answer is simple: in the late 1940s, programming was all the range.
- If you called it programming, your changes of getting a grant drastically increased.
- So we have dynamic programming, quadratic programming, etc.
- All this has nothing to do with programming.
- It is somewhat like now, when many folks processing kilobytes of data call it big data :-(

# 7. An Example of a Linear Programming Problem

- One of the first examples of linear programming was developing meals plan for jails.
- In this case, $v_1, \ldots, v_n$ are amounts of different products: beef, chicken, beans, bread, milk, etc.
- The objective is to minimize cost $\sum\limits_{j=1}^{n} c_j \cdot v_j$.
- The main constraint is that the overall amount of calories should be sufficient: $\sum\limits_{j=1}^{n} a_{1j} \cdot v_j \geq b_1$.
- Here, $a_{1j}$ is calories per pound for the $j$-th product.
- We must also make sure that the folks get:
  - enough proteins $b_2$,
  - enough of different vitamins $b_3, \ldots$,
  - enough of different micro-elements $b_i$, etc.

# 8. Jail Example: Comment

- The solution, by the way, was indeed cheap.
- However, I would not advise students to use it: it does not take taste into account :-(

# 9. How to Solve Linear Programming Problems

- Since linear programming problems are ubiquitous, people have been trying to solve them.
- It started with a simple mathematical analysis.
- Each constraint $\sum_{j=1}^{n} a_{ij} \cdot v_j \le b_i$ determines a half-space.
- A half-space $H$ is a convex set: if $h \in H$ and $h' \in H$, then the whole straight line segment is in $H$:

$$\alpha \cdot h + (1 - \alpha) \cdot h' \in H \text{ for all } \alpha \in (0, 1).$$

- The set of all $v = (v_1, \ldots, v_n)$ that satisfy all the constraints is an intersection of several half-spaces.
- This intersection is thus also convex: a convex polytope.
- On each segment, a linear function is linear.

## 10. Solving Linear Programming (cont-d)

- The maximum of a linear function of a segment is attained at the endpoints.
- So, in our problem, the maximum of a linear function is attained at one of the vertices.
- A vertex is where $n$ of $m$ constraints are equalities.
- Once we know which constraints are equalities, to find $v$, we solve a system of linear equations $\sum a_{ij} \cdot v_j = b_i$.
- There are efficient algorithms for solving such systems; e.g., Gauss elimination takes time $O(n^3)$.
- Problem: there are exponentially many size-$n$ subsets.
- Idea: start with any vertex, and then replace one of the constraints so as to increase the objective function.
- This idea – known as *simplex method* – leads to a very efficient algorithm which is still used.

# 11. Simplex Method (cont-d)

- Its authors, Leonid Kantorovich and Tjalling C. Koopmans, received 1975 Nobel Prize in Economics.
- Problem: sometimes, this algorithm requires exponential time.
- Interestingly, its average computation time is good.
- However, this good time assumes that all the coefficients $a_{ij}$, $b_i$, and $c_j$ are independent.
- In contrast, in practice, they are often strongly correlated.
- As a result, exponential time occurs frequently in practice.

## 12. Can We Reduce Computation Time?

- The authors of the notion of NP-hardness thought that linear programming is NP-hard.
- The theoretical breakthrough was achieved in 1979 by Leonid Khachiyan's polynomial-time algorithm.
- His main idea was to enclose the convex polytope $P$ by an ellipsoid.
- Why ellipsoids?
- The class of problems remains the same if we have a linear change of variables: $v_j \to v'_j = \sum\limits_{j'=1}^{n} d_{jj'} \cdot v_j$.
- The simplest domain is a sphere.
- If we apply different linear transformations to a sphere, we get ellipsoids.

# 13. Khachiyan's Algorithm and Beyond

- We take a known point $p$ satisfying all the constraints.
- Then, we divide the ellipsoid in two by a hyperplane containing $p$ and $\perp c = (c_1, \ldots, c_n)$.
- In the upper half-ellipsoid – where the values of the objective function are higher.
- So, we enclosed this half-ellipsoid a (smaller) ellipsoid, etc.
- While Khachiyan's algorithm was theoretically good, in practice, it was very inefficient.
- In 1984, Narendra Karmarkar proposed a practically efficient version of this algorithm.

## 14. Karmarkar's Algorithm (cont-d)

- His idea is that the class of ellipsoids is also invariant with respect to projective transformations.
- Examples are projections producing a 2-D map of a 3-D Earth.
- So, if we know a point in $P$, we first perform a projective transformation that makes $P$ the ellipsoid's center.
- Only then we bisect.
- Karmarkar's algorithm – and its improvements – are still widely used in practice.
- But it still takes too long.

# 15. Why Cannot We Decrease Computation Time by Parallelization

- When it takes too long for a person to perform a task, this person asks for help.
- When several people work on different parts of the task, the task gets done faster.
- Similarly, many computations become faster if we use several processors working in parallel.
- Unfortunately, this idea does not work for linear programming.
- It has been proven that linear programming is the worst possible problem for parallelization.
- Such problems are known as P-hard.
- So, we cannot just parallelize the existing algorithms: we need new algorithms to speed up computations.

# 16. Let Us Go Back to Constraint Satisfaction

- To find out what to do let us go back and consider constraint satisfaction in general.
- In real life, we often have many constraints that we want to be satisfied.
- For example, in economics, we want:
  - inflation not larger than some reasonably small threshold,
  - unemployment not larger than some small number,
  - growth larger than some minimal amount, etc.
- In practice, several of these constraints are usually not satisfied.
- So, what do we do?
- We select a constraint that is the farther from satisfaction, and concentrate on it.

## 17. Constraint Satisfaction (cont-d)

- For example, if inflation is high, we decrease the money supply.
- Then, inflation goes down, but unemployment goes up and growth stagnates.
- If stagnation becomes the main issue, we concentrate on growth and stimulate economy, etc.
- The same strategy is often used in general:
- We start with some alternative $v^{(0)}$ – which, in general, does not satisfy all the constraints.
- Then, we pick a constraint $C$.
- We find an alternative $v^{(1)}$ which is the closest to $v^{(0)}$ among those that satisfy this constraint:

$$d(v^{(1)}, v^{(0)}) = \min_{x \in C} d(v, v^{(0)}).$$

## 18. Constraint Satisfaction (cont-d)

- After that, we pick another constraint $C'$.
- We find an alternative $v^{(2)}$ which is the closest to $v^{(1)}$ among those that satisfy this constraint:

$$d(v^{(2)}, v^{(1)}) = \min_{x \in C'} d(v, v^{(1)}), \text{ etc.}$$

- In many cases, this process converges either in finitely many steps or in the limit.
- As a result, we get an alternative $v$ that satisfies all the constraints.
- Problem: convergence is often slow.
- For example, for linear programming, this often requires exponential time.

## 19. Sergei Chubanov's Idea

- We want to have $g_i(x_1, \ldots, x_n) \le t_i$ for all $i$.
- If all these inequalities hold, then, for any $\alpha_i \ge 0$, we have $g(x_1, \ldots, x_n) \le t$, where

$$g(x_1, \ldots, x_n) = \sum_{i=1}^{m} \alpha_i \cdot g_i(x_1, \ldots, x_n) \text{ and } t = \sum_{i=1}^{m} \alpha_i \cdot t_i.$$

- These new constraints are known as *derivative constraints*.
- Sergei Chubanov's idea: use general idea, but:
  - instead of cycling through *original* constraints,
  - let us generate *new* derivative constraints every time,
  - here, $\alpha_i$ selected so as to speed up convergence.

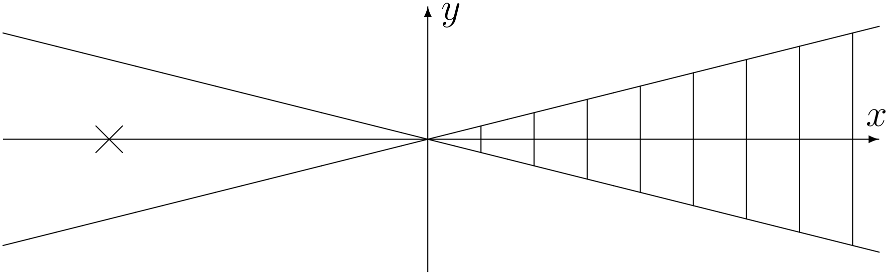
## 20. Chubanov's Idea (cont-d)

- Chubanov has shown that:
  - by appropriately selecting derivative constraints,
  - we can get a polynomial-time algorithm.
- To find $\alpha_i$, we – approximately – solve an optimization problem on each step.
- This is rather technical, not easy to explain.
- But what is easy to explain is why this often drastically speed up convergence.
- Suppose that we want to satisfy two constraints

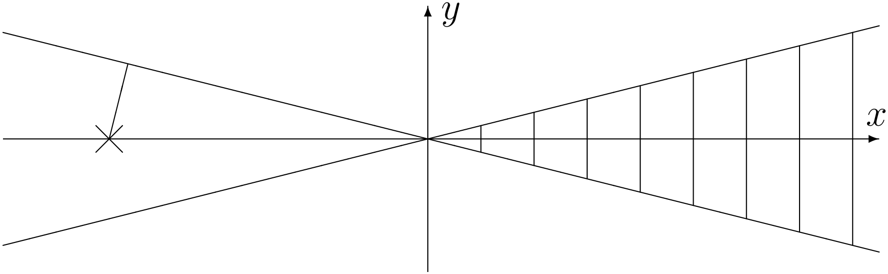
$$y \le \varepsilon \cdot x \text{ and } -y \le \varepsilon \cdot x \text{ for some small } \varepsilon > 0.$$

- Let us start with a point $(-1, 0)$.

# 21. Chubanov's Idea: Example

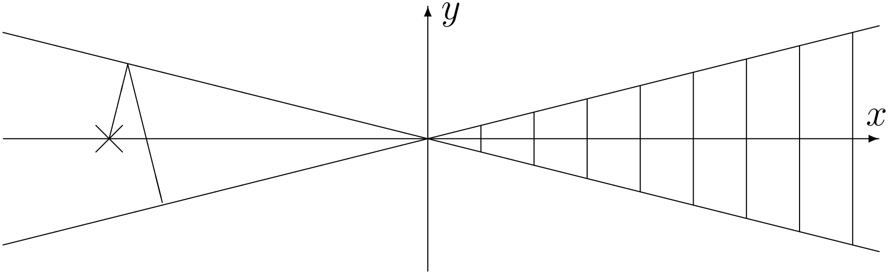



In the traditional constraint satisfaction algorithm, we first "project" onto one of the constraints:

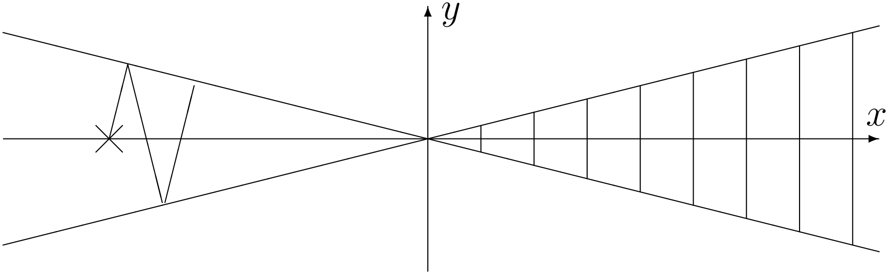


# 22. Example (cont-d)

Then we project onto another constraint:

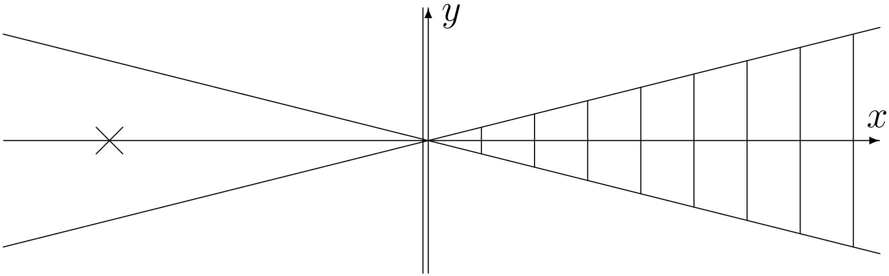



Then onto another one, etc.:



# 23. Example (cont-d)

- For small $\varepsilon$, in the traditional approach, we get a very slow convergence to the desired area.
- In Chubanov's approach, we come up with a derivative constraint $0 \le x$:




- The corresponding projection bring us immediately into a point $(0,0)$ satisfying both constraints:

## 24. Example (cont-d)

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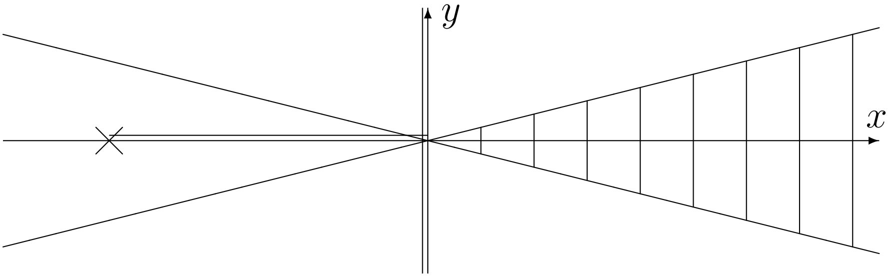


# 25. Why Chubanov's Algorithm Works? Why Other Algorithms Work?

- For LP, there are symmetries behind efficient algorithms.
- This makes sense.
- Indeed, let us assume that there are natural symmetries $T$ on the set of alternatives $A$.
- In this case:
  - alternatives are algorithms, and
  - symmetries are, e.g., linear transformations that keep the problem unchanged.
- On the set $A$, we have a preference relation $\preceq$.
- This relation should be reflexive and transitive – i.e., it should be a (partial) pre-order.

# 26. Why Algorithms Work (cont-d)

- The relation $\preceq$ should be $T$-invariant: if $a \preceq a'$, then $T(a) \preceq T(a')$.
- If several alternatives are the best, this means that we can use this non-uniqueness to optimize something else.
- For example:
  - if several algorithms have the same worst-case complexity $w$,
  - we can select the one with the best average-time $t$.
- In other words, we will use a new preference relation:

$$a \preceq_{\text{new}} a' \Leftrightarrow (w(a') < w(a) \vee (w(a') = w(a) \,\&\, t(a') < t(a)).$$

- If we still have several best alternatives, we can optimize something else, etc.
- At the end, we get a *final* preference relation for which only one optimal alternative is the best.

## 27. Why Algorithms Work (cont-d)

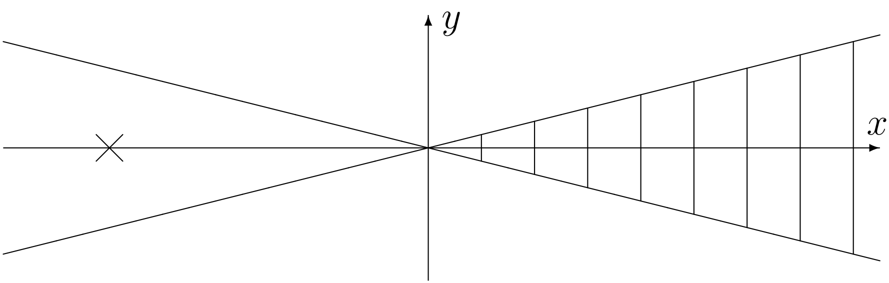
- One can prove that this optimal alternative $a_{\rm opt}$ is itself $T$-invariant.
- Indeed, $a_{\rm opt}$ is better than any other: $a \preceq a_{\rm opt}$.
- In particular, for each $a$, we have $T^{-1}(a) \preceq a_{\rm opt}$.
- Since $\preceq$ is $T$-invariant, we conclude that

$$T(T^{-1}(a)) = a \preceq T(a_{\rm opt}) \text{ for all } a.$$

- Thus, $T(a_{\rm opt})$ is also optimal.
- However, since the preference relation is final, there is only one optimal alternative, thus $T(a_{\rm opt}) = a_{\rm opt}$.

## 28. Back to Chubanov's Algorithm

- From this viewpoint:
  - if it turned out that Chubanov's algorithm is invariant relative to some natural symmetries,
  - this will be a good indication that it is indeed optimal in some sense.
- Let us look at the above example:
  - constraints $y \le \varepsilon \cdot x$ and $-y \le \varepsilon \cdot x$ with
  - initial approximation $x^{(0)} = -1$ and $y^{(0)} = 0$.



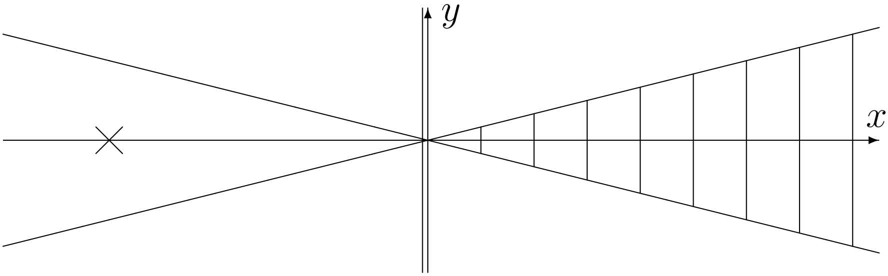
# 29. Chubanov's Algorithm (cont-d)

- This configuration is invariant with respect to $y \to -y$.
- However, in the traditional constraint satisfaction algorithm, this symmetry is violated:
  - we either start with the first constraint,
  - or we start with the second constraint.
- In Chubanov's algorithm, instead, we find $\alpha_i \geq 0$ to form a symmetric derivative constraint:

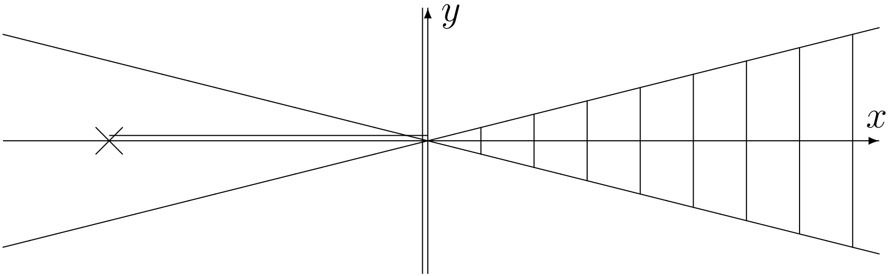
$$\alpha_1 \cdot y + \alpha_2 \cdot (-y) \leq \alpha_1 \cdot \varepsilon \cdot x + \alpha_2 \cdot \varepsilon \cdot x.$$

- This constraint is invariant w.r.t. $y \to -y$ if and only if $\alpha_1 = \alpha_2$.
- Then, we get $0 \leq 2\alpha_i \cdot \varepsilon \cdot x$, i.e., $0 \leq x$.

# 30. Chubanov's Algorithm (cont-d)




The closest point satisfying this derivative constraint is $(0, 0)$ – so Chubanov's algorithm is symmetric!



## 31. Acknowledgments

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