

Multi-Objective Optimization under Positivity Constraints, with a Meteorological Example

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1. Introduction

- *Challenge*: in many remote areas, meteorological sensor coverage is sparse.
- *Desirable*: design sensor networks that provide the largest amount of useful information within a given budget.
- *Difficulty*: because of the huge uncertainty, this problem is very difficult even to formulate in precise terms.
- *First aspect* of the problem: how to best distribute the sensors over the large area.
- *Status*: reasonable solutions exist for this aspect.
- *Second aspect* of the problem: what is the best location of each sensor in the corresponding zone.
- *This talk*: will focus on this aspect of the sensor placement problem.

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2. Outline

- *Case study*: meteorological tower.
- *This case* is an example of multi-criteria optimization, when we need to maximize several objectives x_1, \dots, x_n .
- *Traditional approach* to multi-objective optimization: maximize a weighted combination $\sum_{i=1}^n w_i \cdot x_i$.
- *Specifics of our case*: constraints $x_i > x_i^{(0)}$ or $x_i < x_i^{(0)}$.
- *Equiv.*: $y_i > 0$, where $y_i \stackrel{\text{def}}{=} x_i - x_i^{(0)}$ or $y_i = x_i^{(0)} - x_i$.
- *Limitations* of using the traditional approach under constraints.
- *Scale invariance*: a brief description.
- *Main result*: scale invariance leads to a new approach: maximize $\sum_{i=1}^n w_i \cdot \ln(y_i) = \sum_{i=1}^n w_i \cdot \ln \left| x_i - x_i^{(0)} \right|$.

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3. Case Study

- *Objective:* select the best location of a sophisticated multi-sensor meteorological tower.
- *Constraints:* we have several criteria to satisfy.
- *Example:* the station should not be located too close to a road.
- *Motivation:* the gas flux generated by the cars do not influence our measurements of atmospheric fluxes.
- *Formalization:* the distance x_1 to the road should be larger than a threshold t_1 : $x_1 > t_1$, or $y_1 \stackrel{\text{def}}{=} x_1 - t_1 > 0$.
- *Example:* the inclination x_2 at the tower's location should be smaller than a threshold t_2 : $x_2 < t_2$.
- *Motivation:* otherwise, the flux determined by this inclination and not by atmospheric processes.

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4. General Case

- *In general*: we have several differences y_1, \dots, y_n all of which have to be non-negative.
- For each of the differences y_i , the larger its value, the better.
- Our problem is a typical setting for *multi-criteria optimization*.
- A most widely used approach to multi-criteria optimization is *weighted average*, where
 - we assign weights $w_1, \dots, w_n > 0$ to different criteria y_i and
 - select an alternative for which the weighted average

$$w_1 \cdot y_1 + \dots + w_n \cdot y_n$$

attains the largest possible value.

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5. Limitations of the Weighted Average Approach

- *In general:* the weighted average approach often leads to reasonable solutions of the multi-criteria problem.
- *In our problem:* we have an additional requirement – that all the values y_i must be positive. So:
 - when selecting an alternative with the largest possible value of the weighted average,
 - we must only compare solutions with $y_i > 0$.
- *We will show:* under the requirement $y_i > 0$, the weighted average approach is not fully satisfactory.
- *Conclusion:* we need to find a more adequate solution.

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6. Limitations of the Weighted Average Approach: Details

- The values y_i come from measurements, and measurements are never absolutely accurate.
- The results \tilde{y}_i of the measurements are not exactly equal to the actual (unknown) values y_i .
- *If*: for some alternative $y = (y_1, \dots, y_n)$
 - we measure the values y_i with higher and higher accuracy and,
 - based on the measurement results \tilde{y}_i , we conclude that y is better than some other alternative y' .
- *Then*: we expect that the actual alternative y is indeed better than y' (or at least of the same quality).
- Otherwise, we will not be able to make any meaningful conclusions based on real-life measurements.

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7. The Above Natural Requirement Is Not Always Satisfied for Weighted Average

- *Simplest case*: two criteria y_1 and y_2 , w/weights $w_i > 0$.
- If $y_1, y_2, y'_1, y'_2 > 0$, and $w_1 \cdot y_1 + w_2 \cdot y_2 > w_1 \cdot y'_1 + w_2 \cdot y'_2$, then $y = (y_1, y_2) \succ y' = (y'_1, y'_2)$.
- If $y_1 > 0, y_2 > 0$, and at least one of the values y'_1 and y'_2 is non-positive, then $y = (y_1, y_2) \succ y' = (y'_1, y'_2)$.
- Let us consider, for every $\varepsilon > 0$, the tuple $y(\varepsilon) \stackrel{\text{def}}{=} (\varepsilon, 1 + w_1/w_2)$, and $y' = (1, 1)$.
- In this case, for every $\varepsilon > 0$, we have
$$w_1 \cdot y_1(\varepsilon) + w_2 \cdot y_2(\varepsilon) = w_1 \cdot \varepsilon + w_2 + w_2 \cdot \frac{w_1}{w_2} = w_1 \cdot (1 + \varepsilon) + w_2$$
and $w_1 \cdot y'_1 + w_2 \cdot y'_2 = w_1 + w_2$, hence $y(\varepsilon) \succ y'$.
- However, in the limit $\varepsilon \rightarrow 0$, we have $y(0) = \left(0, 1 + \frac{w_1}{w_2}\right)$, with $y(0)_1 = 0$ and thus, $y(0) \prec y'$.

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8. Heuristic Idea Motivated by Fuzzy Logic

- *Problem*: the first criterion must be satisfied *and* the second criterion must be satisfied, ...
- *Fuzzy logic approach*:
 - First, we estimate the degrees d_1, \dots, d_n to which each of the constraints is satisfied.
 - Then, we use a t-norm (fuzzy analogue of “and”) to combine these degrees into a single degree d .
- *Simplest membership functions*: triangular, for which $d_i(y_i) = k_i \cdot y_i$, with $k_i > 0$ (when $y_i > 0$).
- *Selecting a t-norm*: the simplest is min, but it is not smooth hence tough to optimize; next simplest is $a \cdot b$.
- *Result*: maximize $d = \prod_{i=1}^n (k_i \cdot y_i) \Leftrightarrow \text{maximize } \prod_{i=1}^n y_i$.
- *This approach is indeed better than weighted average*: e.g., if $y'(\varepsilon) \succ y$ and $y'(\varepsilon) \rightarrow y'(0)$, then $y'(0) \succeq y$.

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9. Natural Next Idea: Use Hedges

- *Idea*: different criteria have different importance:
 - for some criteria, it is sufficient to have them somewhat satisfied;
 - for others, they must be very satisfied.
- So, instead of combining degrees d_i , we combined hedged degrees $h_i(d_i)$.
- *Natural requirement*: e.g., “very ($a \& b$)” should mean the same as “very a and very b ”.
- Thus, $h(a \cdot b) = h(a) \cdot h(b)$ and hence, $h(a) = a^\alpha$.
- *Conclusion*: we combine $h(d_i) = d_i^{\alpha_i}$, i.e., we optimize the product $\prod_{i=1}^n y_i^{\alpha_i}$.
- *What we prove*: this fuzzy-motivated expression is the *only* expression that satisfies reasonable properties.

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10. Towards a Precise Description

- Each alternative is characterized by a tuple of n positive values $y = (y_1, \dots, y_n)$.
- Thus, the set of all alternatives is the set $(R^+)^n$ of all the tuples of positive numbers.
- For each two alternatives y and y' , we want to tell whether
 - y is better than y' (we will denote it by $y \succ y'$ or $y' \prec y$),
 - or y' is better than y ($y' \succ y$),
 - or y and y' are equally good ($y' \sim y$).
- *Natural requirement*: if y is better than y' and y' is better than y'' , then y is better than y'' .
- The relation \succ must be transitive.

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11. Towards a Precise Description (cont-d)

- *Reminder*: the relation \succ must be transitive.
- Similarly, the relation \sim must be transitive, symmetric, and reflexive ($y \sim y$), i.e., be an *equivalence relation*.
- *An alternative description*: a transitive pre-ordering relation $a \succeq b \Leftrightarrow (a \succ b \vee a \sim b)$ s.t. $a \succeq b \vee b \succeq a$.
- Then, $a \sim b \Leftrightarrow (a \succeq b) \& (b \succeq a)$, and

$$a \succ b \Leftrightarrow (a \succeq b) \& (b \not\succeq a).$$

- *Additional requirement*:
 - if each criterion is better,
 - then the alternative is better as well.
- *Formalization*: if $y_i > y'_i$ for all i , then $y \succ y'$.

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12. Scale Invariance: Motivation

- *Fact:* quantities y_i describe completely different physical notions, measured in completely different units.
- *Examples:* wind velocities measured in m/s, km/h, mi/h; elevations in m, km, ft.
- Each of these quantities can be described in many different units.
- A priori, we do not know which units match each other.
- Units used for measuring different quantities may not be exactly matched.
- It is reasonable to require that:
 - if we simply change the units in which we measure each of the corresponding n quantities,
 - the relations \succ and \sim between the alternatives $y = (y_1, \dots, y_n)$ and $y' = (y'_1, \dots, y'_n)$ do not change.

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13. Scale Invariance: Towards a Precise Description

- *Situation*: we replace:
 - a unit in which we measure a certain quantity q
 - by a new measuring unit which is $\lambda > 0$ times smaller.
- *Result*: the numerical values of this quantity increase by a factor of λ : $q \rightarrow \lambda \cdot q$.
- *Example*: 1 cm is $\lambda = 100$ times smaller than 1 m, so the length $q = 2$ becomes $\lambda \cdot q = 2 \cdot 100 = 200$ cm.
- Then, scale-invariance means that for all $y, y' \in (R^+)^n$ and for all $\lambda_i > 0$, we have
 - $y = (y_1, \dots, y_n) \succ y' = (y'_1, \dots, y'_n)$ implies $(\lambda_1 \cdot y_1, \dots, \lambda_n \cdot y_n) \succ (\lambda_1 \cdot y'_1, \dots, \lambda_n \cdot y'_n)$,
 - $y = (y_1, \dots, y_n) \sim y' = (y'_1, \dots, y'_n)$ implies $(\lambda_1 \cdot y_1, \dots, \lambda_n \cdot y_n) \sim (\lambda_1 \cdot y'_1, \dots, \lambda_n \cdot y'_n)$.

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14. Formal Description

- By a *total pre-ordering relation* on a set Y , we mean
 - a pair of a transitive relation \succ and an equivalence relation \sim for which,
 - for every $y, y' \in Y$, exactly one of the following relations hold: $y \succ y'$, $y' \succ y$, or $y \sim y'$.
- We say that a total pre-ordering is *non-trivial* if there exist y and y' for which $y \succ y'$.
- We say that a total pre-ordering relation on $(R^+)^n$ is:
 - *monotonic* if $y'_i > y_i$ for all i implies $y' \succ y$;
 - *continuous* if
 - * whenever we have a sequence $y^{(k)}$ of tuples for which $y^{(k)} \succeq y'$ for some tuple y' , and
 - * the sequence $y^{(k)}$ tends to a limit y ,
 - * then $y \succeq y'$.

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15. Main Result

Theorem. *Every non-trivial monotonic scale-inv. continuous total pre-ordering relation on $(R^+)^n$ has the form:*

$$y' = (y'_1, \dots, y'_n) \succ y = (y_1, \dots, y_n) \Leftrightarrow \prod_{i=1}^n (y'_i)^{\alpha_i} > \prod_{i=1}^n y_i^{\alpha_i};$$

$$y' = (y'_1, \dots, y'_n) \sim y = (y_1, \dots, y_n) \Leftrightarrow \prod_{i=1}^n (y'_i)^{\alpha_i} = \prod_{i=1}^n y_i^{\alpha_i},$$

for some constants $\alpha_i > 0$.

Comment: Vice versa,

- for each set of values $\alpha_1 > 0, \dots, \alpha_n > 0$,
- the above formulas define a monotonic scale-invariant continuous pre-ordering relation on $(R^+)^n$.

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16. Practical Conclusion

- *Situation:*
 - we need to select an alternative;
 - each alternative is characterized by characteristics y_1, \dots, y_n .
- *Traditional approach:*
 - we assign the weights w_i to different characteristics;
 - we select the alternative with the largest value of
$$\sum_{i=1}^n w_i \cdot y_i.$$
- *New result:* it is better to select an alternative with the largest value of
$$\prod_{i=1}^n y_i^{w_i}.$$
- *Equivalent reformulation:* select an alternative with the largest value of
$$\sum_{i=1}^n w_i \cdot \ln(y_i).$$

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17. Acknowledgments

This work was supported in part by:

- by National Science Foundation grants HRD-0734825 and DUE-0926721,
- by Grant 1 T36 GM078000-01 from the National Institutes of Health, and
- by Grant 5015 from the Science and Technology Centre in Ukraine (STCU), funded by European Union.

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18. Proof: Part 1

- Due to scale-invariance, for every $y_1, \dots, y_n, y'_1, \dots, y'_n$, we can take $\lambda_i = \frac{1}{y_i}$ and conclude that

$$(y'_1, \dots, y'_n) \sim (y_1, \dots, y_n) \Leftrightarrow \left(\frac{y'_1}{y_1}, \dots, \frac{y'_n}{y_n} \right) \sim (1, \dots, 1).$$

- Thus, to describe the equivalence relation \sim , it is sufficient to describe $\{z = (z_1, \dots, z_n) : z \sim (1, \dots, 1)\}$.
- Similarly,

$$(y'_1, \dots, y'_n) \succ (y_1, \dots, y_n) \Leftrightarrow \left(\frac{y'_1}{y_1}, \dots, \frac{y'_n}{y_n} \right) \succ (1, \dots, 1).$$

- Thus, to describe the ordering relation \succ , it is sufficient to describe the set $\{z = (z_1, \dots, z_n) : z \succ (1, \dots, 1)\}$.
- Similarly, it is also sufficient to describe the set

$$\{z = (z_1, \dots, z_n) : (1, \dots, 1) \succ z\}.$$

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19. Proof: Part 2

- *To simplify:* take logarithms $Y_i = \ln(y_i)$, and sets
$$S_{\sim} = \{Z : z = (\exp(Z_1), \dots, \exp(Z_n)) \sim (1, \dots, 1)\},$$
$$S_{\succ} = \{Z : z = (\exp(Z_1), \dots, \exp(Z_n)) \succ (1, \dots, 1)\};$$
$$S_{\prec} = \{Z : (1, \dots, 1) \succ z = (\exp(Z_1), \dots, \exp(Z_n))\}.$$
- Since the pre-ordering relation is total, for Z , either $Z \in S_{\sim}$ or $Z \in S_{\succ}$ or $Z \in S_{\prec}$.
- *Lemma:* S_{\sim} is closed under addition:
 - $Z \in S_{\sim}$ means $(\exp(Z_1), \dots, \exp(Z_n)) \sim (1, \dots, 1)$;
 - due to scale-invariance, we have
$$(\exp(Z_1 + Z'_1), \dots) = (\exp(Z_1) \cdot \exp(Z'_1), \dots) \sim (\exp(Z'_1), \dots);$$
 - also, $Z' \in S_{\sim}$ means $(\exp(Z'_1), \dots) \sim (1, \dots, 1)$;
 - since \sim is transitive,
$$(\exp(Z_1 + Z'_1), \dots) \sim (1, \dots) \text{ so } Z + Z' \in S_{\sim}.$$

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20. Proof: Part 3

- *Reminder:* the set S_{\sim} is closed under addition;
- Similarly, S_{\prec} and S_{\succ} are closed under addition.
- *Conclusion:* for every integer $q > 0$:
 - if $Z \in S_{\sim}$, then $q \cdot Z \in S_{\sim}$;
 - if $Z \in S_{\succ}$, then $q \cdot Z \in S_{\succ}$;
 - if $Z \in S_{\prec}$, then $q \cdot Z \in S_{\prec}$.
- Thus, if $Z \in S_{\sim}$ and $q \in \mathbb{N}$, then $(1/q) \cdot Z \in S_{\sim}$.
- We can also prove that S_{\sim} is closed under $Z \rightarrow -Z$:
 - $Z = (Z_1, \dots) \in S_{\sim}$ means $(\exp(Z_1), \dots) \sim (1, \dots)$;
 - by scale invariance, $(1, \dots) \sim (\exp(-Z_1), \dots)$, i.e., $-Z \in S_{\sim}$.
- Similarly, $Z \in S_{\succ} \Leftrightarrow -Z \in S_{\prec}$.
- So $Z \in S_{\sim} \Rightarrow (p/q) \cdot Z \in S_{\sim}$; in the limit, $x \cdot Z \in S_{\sim}$.

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21. Proof: Final Part

- *Reminder:* S_{\sim} is closed under addition and multiplication by a scalar, so it is a linear space.
- *Fact:* S_{\sim} cannot have full dimension n , since then all alternatives will be equivalent to each other.
- *Fact:* S_{\sim} cannot have dimension $< n - 1$, since then:
 - we can select an arbitrary $Z \in S_{\prec}$;
 - connect it w/ $-Z \in S_{\succ}$ by a path γ that avoids S_{\sim} ;
 - due to closeness, $\exists \gamma(t^*)$ in the limit of S_{\succ} and S_{\prec} ;
 - thus, $\gamma(t^*) \in S_{\sim}$ – a contradiction.
- Every $(n - 1)$ -dim lin. space has the form $\sum_{i=1}^n \alpha_i \cdot Y_i = 0$.
- Thus, $Y \in S_{\succ} \Leftrightarrow \sum \alpha_i \cdot Y_i > 0$, and

$$y \succ y' \Leftrightarrow \sum \alpha_i \cdot \ln(y_i/y'_i) > 0 \Leftrightarrow \prod y_i^{\alpha_i} > \prod y'_i{}^{\alpha_i}.$$

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