Towards a More Natural Proof of Metrization Theorem for Space-Times

Vladik Kreinovich and Olga Kosheleva

University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
contact email vladik@utep.edu

Urysohn's Lemma and . . .
Extension to Space- . . .
Space-Time Analogs . . .
Towards a Fuzzy . . .
Degree Interpretation . . .
Towards a More . . .
Let Us Use the . . .
Towards a More . . .
Towards a More . . .

1. Urysohn's Lemma and Urysohn's Metrization Theorem: Reminder

- *Who, when:* early 1920s, Pavel Urysohn.
- *Claim for fame:* Urysohn's Lemma is "first non-trivial result of point set topology".
- *Condition:* X is a normal topological space X, A and B are disjoint closed sets.
- *Conclusion:* there exists $f : X \to [0,1]$ s.t. $f(A) = \{0\}$ and $f(B) = \{1\}$.
- *Reminder:* normal means that every two disjoint closed sets have disjoint open neighborhoods.
- *Application:* every normal space with countable base is metrizable.
- *Comment:* actually, every regular Hausdorff space with countable base is metrizable.

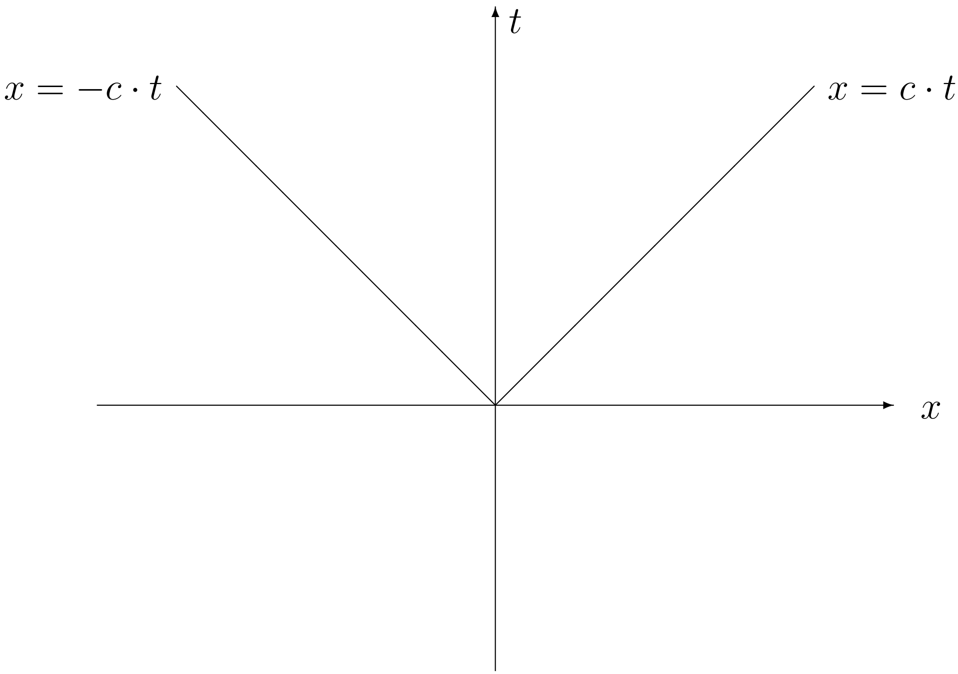
Urysohn's Lemma and . . .
Extension to Space- . . .
Space-Time Analogs . . .
Towards a Fuzzy . . .
Degree Interpretation . . .
Towards a More . . .
Let Us Use the . . .
Towards a More . . .
Towards a More . . .

2. Extension to Space-Times: Urysohn's Problem

- *Fact:* a few years before that, in 1919, Einstein's GRT has been experimentally confirmed.
- *Corresponding structure:* topological space with an order (casuality).
- *Urysohn's problem:* extend his lemma and metrization theorem to (causality-)ordered topological spaces.
- *Tragic turn of events:* Urysohn died in 1924.
- *Follow up:* Urysohn's student Vadim Efremovich; Efremovich's student Revolt Pimenov; Pimenov's students.
- *Other researchers:* H. Busemann (US), E. Kronheimer and R. Penrose (UK).
- *Result:* by the 1970s, space-time versions of Uryson's lemma and metrization theorem have been proven.

Urysohn's Lemma and . . .
Extension to Space- . . .
Space-Time Analogs . . .
Towards a Fuzzy . . .
Degree Interpretation . . .
Towards a More . . .
Let Us Use the . . .
Towards a More . . .
Towards a More . . .

3. Causality: A Reminder



Urysohn's Lemma and ...
Extension to Space- ...
Space-Time Analogs ...
Towards a Fuzzy ...
Degree Interpretation ...
Towards a More ...
Let Us Use the ...
Towards a More ...
Towards a More ...

4. Space-Time Metrization Results: A Challenge

- *Objective:* come up with useful applications to physics.
- *Conclusion:* we need proofs that directly follow from the analysis of the main notions and ideas.
- *Fact:* the original 1970s proofs look like the use of clever tricks.
- *Conclusion:* we must make these proofs more natural.
- *How:* we use Zadeh's ideas of applying fuzzy to causality.
- *We show:* that fuzzy logic indeed leads to such more natural proofs.
- *Not yet:* we are still far away from practical applications.
- *We believe* that our result has brought us one step closer to these future applications.

Urysohn's Lemma and . . .
Extension to Space- . . .
Space-Time Analogs . . .
Towards a Fuzzy . . .
Degree Interpretation . . .
Towards a More . . .
Let Us Use the . . .
Towards a More . . .
Towards a More . . .

5. Space-Time Models: Reminder

- *Theoretical relation:* (transitive) causality $a \preceq b$.
- *Problem:* events are not located exactly: $\widetilde{a} \approx a$, $\widetilde{b} \approx b$.
- *Practical relation:* kinematic casuality $a \prec b$.
- *Meaning:* every event in some small neighborhood of b causally follows a, i.e., $b \in \text{Int}(a^+)$.
- *Properties of $\prec$:* $\prec$ is transitive; $a \not\prec a$;

$$\forall a\, \exists \underline{a}, \overline{a}\, (\underline{a} \prec a \prec \overline{a}); \quad a \prec b \Rightarrow \exists c\, (a \prec c \prec b);$$

$$a \prec b, c \Rightarrow \exists d\, (a \prec d \prec b, c); \quad b, c \prec a \Rightarrow \exists d\, (b, c \prec d \prec a).$$

- *Alexandrov topology:* with intervals as the base:

$$(a, b) \stackrel{\text{def}}{=} \{c : a \prec c \prec b\}.$$

- *Description of causality:* $a \preceq b \stackrel{\text{def}}{\equiv} b \in \overline{a^+}$.
- *Additional property:* $b \in \overline{a^+} \Leftrightarrow a \in \overline{b^-}$.

Urysohn's Lemma and . . .
Extension to Space- . . .
Space-Time Analogs . . .
Towards a Fuzzy . . .
Degree Interpretation . . .
Towards a More . . .
Let Us Use the . . .
Towards a More . . .
Towards a More . . .

6. Space-Time Analog of a Metric

- *Traditional metric:* a function $\rho : X \times X \to R_0^+$ s.t.

$$\rho(a,b) = 0 \Leftrightarrow a = b;$$

$$\rho(a,b) = \rho(b,a);$$

$$\rho(a,c) \le \rho(a,b) + \rho(b,c).$$

- *Physical meaning:* the length of the shortest path between a and b.
- *Kinematic metric:* a function $\tau : X \times X \to R_0^+$ s.t.

$$\tau(a,b) > 0 \Leftrightarrow a \prec b;$$

$$a \prec b \prec c \Rightarrow \tau(a,c) \ge \tau(a,b) + \tau(b,c).$$

- *Physical meaning:* the longest (= proper) time from event a to event b.
- *Explanation:* when we speed up, time slows down.

Urysohn's Lemma and . . .
Extension to Space- . . .
Space-Time Analogs . . .
Towards a Fuzzy . . .
Degree Interpretation . . .
Towards a More . . .
Let Us Use the . . .
Towards a More . . .
Towards a More . . .

7. Space-Time Analogs of Urysohn's Lemma and Metrization Theorem

- *Main condition:* the kinematic space is *separable*, i.e., there exists a countable dense set $\{x_1, x_2, \ldots, x_n, \ldots\}$.
- *Condition of the lemma:* X is separable, and $a \prec b$.
- *Lemma:* $\exists$ a cont. $\preceq$-increasing f-n $f_{(a,b)} : X \to [0, 1]$ s.t. $f_{(a,b)}(x) = 0$ for $a \not\prec x$ and $f_{(a,b)}(x) = 1$ for $b \preceq x$.
- *Relation to the original Urysohn's lemma:* $f_{(a,b)}$ separates disjoint closed sets $-a^+$ and $\overline{b^+}$.
- *Condition of the theorem:* $(X, \prec)$ is a separable kinematic space.
- *Theorem:* there exists a continuous metric τ which generates the corresponding relation $\prec$.
- *Corollary:* τ also generates the corresponding topology.

Urysohn's Lemma and ...
Extension to Space- ...
Space-Time Analogs ...
Towards a Fuzzy ...
Degree Interpretation ...
Towards a More ...
Let Us Use the ...
Towards a More ...
Towards a More ...

8. How the Space-Time Metrization Theorem Is Proved Now

- *First lemma:* for every x, there exists a $\prec$-monotonic function $f_x : X \to [0,1]$ for which $f_x(b) > 0 \Leftrightarrow x \prec b$.
- *Proof:* $\exists y_i \searrow x$; take $f_x(b) = \sum\limits_{i=1}^{\infty} 2^{-i} \cdot f_{(x,y_i)}(b)$.
- *Second lemma:* for every x, there exists a $\prec$-monotonic function $g_x : X \to [0,1]$ for which $g_x(a) > 0 \Leftrightarrow a \prec x$.
- *Proof:* similar.
- *Resulting metric:* for a countable everywhere dense sequence $\{x_1, x_2, \ldots, x_n, \ldots\}$, take

$$\tau(a,b) = \sum_{i=1}^{\infty} 2^{-i} \cdot \min(g_{x_i}(a), f_{x_i}(b)).$$

Urysohn's Lemma and . . .
Extension to Space- . . .
Space-Time Analogs . . .
Towards a Fuzzy . . .
Degree Interpretation . . .
Towards a More . . .
Let Us Use the . . .
Towards a More . . .
Towards a More . . .

9. Towards a Fuzzy Interpretation of Space-Time Ideas

- *Intuitive idea:* $\tau(a,b)$ is a "degree of causality".
- *Why fuzzy logic:* it was specifically designed to generate and process such degrees.
- *Simplest case:* we only have a lower bound $\underline{x}$ and an upper bound $\overline{x}$ for a quantity x.
- *Description:* the value x belongs to the *interval*

$$[\underline{x}, \overline{x}] \stackrel{\text{def}}{=} \{x : \underline{x} \le x \le \overline{x}\}.$$

- *Space-time case:* we know:
  - an event $\underline{x}$ that influenced x (i.e., that causally precedes x), and
  - an event $\overline{x}$ that was influenced by x (i.e., that causally follows from x).
- *Description:* $[\underline{x}, \overline{x}] \stackrel{\text{def}}{=} \{x : \underline{x} \preceq x \preceq \overline{x}\}.$

Urysohn's Lemma and ...
Extension to Space- ...
Space-Time Analogs ...
Towards a Fuzzy ...
Degree Interpretation ...
Towards a More ...
Let Us Use the ...
Towards a More ...
Towards a More ...

10. Degree Interpretation of Urysohn's Lemma

- *We know:* that the event e is in the interval $[a, b]$.
- *We want to estimate:* the degree $f_{[a,b]}(x)$ to which it is possible that $e \preceq x$.
- *1st property:* if $b \preceq x$, then $e \preceq x$ hence $f_{[a,b]}(x) = 1$.
- *2nd property:* is $a \not\preceq x$, then $e \not\preceq x$ hence $f_{[a,b]}(x) = 0$.
- *Fact:* if $x \preceq x'$ and $e \preceq x$, then, of course, $e \preceq x'$.
- *Thus:* our degree of possibility that $e \preceq x'$ is larger (or equal) than the degree that $e \preceq x$: $f_{[a,b]}(x) \le f_{[a,b]}(x')$.
- *3rd property:* in mathematical terms, this means that the function $f_{[a,b]}(x)$ is $\preceq$-*monotonic.*
- When we change x slightly, our degree $f_{[a,b]}(x)$ should change only slightly, i.e., $f_{[a,b]}$ should be *continuous.*
- These are exactly the properties of a function existing due to the space-time analogue of Urysohn's lemma.

Urysohn's Lemma and . . .
Extension to Space- . . .
Space-Time Analogs . . .
Towards a Fuzzy . . .
Degree Interpretation . . .
Towards a More . . .
Let Us Use the . . .
Towards a More . . .
Towards a More . . .

11. Towards a More Natural Proof of the Lemmas

- *Ideal case:* $x \preceq b \Leftrightarrow b$ is influenced by a signal emitted at the moment x.
- *In practice:* there is always a delay between the decision x to emit the signal and the actual emission y.
- Influence $x \preceq b$ is confirmed if b follows from some event $y \in [x, y_1]$, for a known upper bound y_1.
- By using more and more accurate technologies, we can make this delay smaller and smaller, with y_i such that

$$x \prec \ldots \prec y_3 \prec y_2 \prec y_1 \text{ and } y_i \to x.$$

- Thus, $x \preceq b \Leftrightarrow \exists i (y \prec b \text{ for some } y \in [x, y_i]))$.
- *In practice:* large i may require technology which is not yet available, so

$$\exists i \, ((i \text{ is not too large}) \,\&\, (y \prec b \text{ for some } y \in [x, y_i])).$$

Urysohn's Lemma and . . .
Extension to Space- . . .
Space-Time Analogs . . .
Towards a Fuzzy . . .
Degree Interpretation . . .
Towards a More . . .
Let Us Use the . . .
Towards a More . . .
Towards a More . . .

12. Let Us Use the Simplest Fuzzy Translations

- $\exists i\,((i \text{ is not too large})\,\&\,(y \prec b \text{ for some } y \in [x, y_i]))$.
- *We know:* the degree of belief $f_{[x,y_i]}(b)$ that $y \prec b$ for some $y \in [x, y_i]$.
- *We need* to describe:
  - the degree of belief $N(i)$ that i is not too large,
  - the degree of belief that $y \prec b$ for some $y \in [x, y_i]$,
  - t-norm ("and") and t-conorm ("or") operations $\&$ and $\vee$:

$$d(A), d(B) \to d(A\,\&\,B) \approx d(A)\,\&\,d(B), d(A \vee B) \approx d(A) \vee d(B).$$

- *Our selection:* computationally simplest $a\,\&\,b = a \cdot b$ and $a \vee b = \min(a + b, 1)$ (in our case $a \vee b = a + b$).
- *Comment:* our proof works for other choices as well.
- *Remaining problem:* describe the degrees $N(i)$.

13. Towards a More Natural Proof of Lemma 1

- If (i is not too large) and (j is not too large) then ($i+j$ is not too large).
- *Description:* $N(i+j) = N(i) \,\&\, N(j) = N(i) \cdot N(j)$.
- Thus, we get $N(i) = N(i-1) \cdot N(1)$ and $N(i) = N(1)^i$.
- *Simplest case:* $N(1) = 1/2$, then $N(i) = 2^{-i}$.
- $\exists i\,((i \text{ is not too large}) \,\&\, (y \prec b \text{ for some } y \in [x, y_i]))$.
- *Reminder:* $a \,\&\, b$ is $a \cdot b$; $\exists i\, P(i)$ is $A_1 \vee A_2 \vee \ldots$; $a \vee b$ is $a + b$; and $d(y \prec b \text{ for some } y \in [x, y_i]) = f_{[x,y_i]}(b)$.
- *Conclusion:* $f_x(b) = \sum\limits_{i=1}^{\infty} 2^{-i} \cdot f_{[x,y_i]}(b)$.
- *Easy to prove:* Lemma 1, i.e., f_x is $\preceq$-monotonic and
$$f_x(b) > 0 \Leftrightarrow x \prec b.$$
- *Comment:* Lemma 2 can be similarly proven.

Urysohn's Lemma and . . .
Extension to Space- . . .
Space-Time Analogs . . .
Towards a Fuzzy . . .
Degree Interpretation . . .
Towards a More . . .
Let Us Use the . . .
Towards a More . . .
Towards a More . . .

14. Towards a More Natural Proof of the Metrization Theorem for Space-Times

- *Idea:* $a \preceq b \Leftrightarrow$ a signal emitted at a is detected at b.
- *Direct implementation:* every event directly sends signals to everyone else.
- *Problem:* too much energy needed.
- *Solution:* retransmissions at x_1, x_2, ...:

$$a \prec b \Leftrightarrow \exists i\, ((a \prec x_i)\,\&\,(x_i \prec b)).$$

- *In practice:* $\exists i\, ((i \text{ is not too large})\,\&\,(a \prec x_i))$.
- *Result* of using fuzzy translation:

$$\tau(a,b) = \sum_{i=1}^{\infty} 2^{-i} \cdot g_{x_i}(a) \cdot f_{x_i}(b).$$

- *It is* (relatively) *easy to prove:* that this expression satisfies both properties of the kinematic metric.

Urysohn's Lemma and ...
Extension to Space- ...
Space-Time Analogs ...
Towards a Fuzzy ...
Degree Interpretation ...
Towards a More ...
Let Us Use the ...
Towards a More ...
Towards a More ...

15. Acknowledgments

- This work was supported in part
  - by the National Science Foundation grant HRD-0734825,
  - by Grant 1 T36 GM078000-01 from the National Institutes of Health,
  - by the Max Planck Institut für Mathematik, and
  - by Grant MSM 6198898701 from MŠMT of Czech Republic.
- The authors are thankful to Lotfi A. Zadeh for his inspiring ideas on using fuzzy logic to describe causality.

Urysohn's Lemma and . . .
Extension to Space- . . .
Space-Time Analogs . . .
Towards a Fuzzy . . .
Degree Interpretation . . .
Towards a More . . .
Let Us Use the . . .
Towards a More . . .
Towards a More . . .