Semi-Heuristic Poverty Measures Used by Economists: Justification Motivated by Fuzzy Techniques

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1. How Poverty is Measured Now

- Usually, there is a *poverty threshold* threshold z: a person i with an income x_i is poor $\Leftrightarrow x_i < z$.
- *Media:* measures property by the proportion $F_0 = \dfrac{H}{N}$ of poor people.
- *Limitation:* F_0 does not distinguish between very poor $(x_i \ll z)$ and simply poor.
- To capture this difference, economists use special Foster-Greer-Thorbecke (FGT) property measures:

$$F_1 = \frac{1}{N} \cdot \sum_{i=1}^{H} \left(1 - \frac{x_i}{z}\right) \text{ and } F_2 = \frac{1}{N} \cdot \sum_{i=1}^{H} \left(1 - \frac{x_i}{z}\right)^2.$$

- *Success:* these measures are used to gauge the success of different measures aimed at reducing poverty.
- *Problem:* these measures are semi-heuristic, other measures may be more adequate.

2. Fuzzy Approach to Poverty

- Poverty is a matter of degree: $\mu(0) = 1$, $\mu(z) = 0$.
- Simplest membership function – linear: $\mu(x) = 1 - \frac{x}{z}$.
- The cardinality of a fuzzy set is defined as $\sum\limits_x \mu(x)$.
- Thus, the cardinality of the set of all poor people is

$$\sum_{i=1}^{H} \left(1 - \frac{x_i}{z}\right).$$

- This sum is proportional to F_1.
- For a fuzzy property P, "very P" is usually interpreted as $\mu^2(x)$.
- Thus, the cardinality of the set of all *very* poor people is $\sum\limits_{i=1}^{H} \left(1 - \frac{x_i}{z}\right)^2$; this sum is proportional to F_2.

3. Fuzzy Approach: Conclusion and Limitations

- *Good news:* all three FGT measures F_0, F_1, and F_2 naturally appear in the fuzzy interpretation.
- Each of F_i is the ratio of the number of poor people to the population as a whole:
  - F_0 appears when we consider poverty to be a crisp property;
  - F_1 appears when we take that poverty is a fuzzy property;
  - F_2 appears when we count the number of *very* poor people.
- *Limitation:* the justification depends on a specific choice of linear membership function (and $\mu^2(x)$ for "very").
- *What we do:* we go from an informal to a precise justification.

4. Towards Precise Definitions

- *Natural requirement:*
  - if we know poverty measures, populations, and number of poor in two subareas,
  - then we should be able to compute the property measure for the whole area.
- *Known result:* all such measures have the form $v_f = \sum_{i=1}^{H} f(x_i)$ for some $f(x)$.
- *Fact:* different measures describe different aspects of poverty.
- *So:* we want to select k measures $\sum_{i=1}^{H} f_j(x_i)$, $j = 1, \ldots, k$.

5. Independence

- *In principle:* based on two measures $f_1(x)$ and $f_2(x)$, we can form a new measure
$$f(x) = \frac{f_1(x) + f_2(x)}{2}.$$
- *In this case:*
  - once we know $v_{f_1} = \sum\limits_{i=1}^{H} f_1(x_i)$ and $v_{f_2} = \sum\limits_{i=1}^{H} f_2(x_i)$,
  - we can reconstruct $v_f = \sum\limits_{i=1}^{H} f(x_i)$ as
$$v_f = \frac{v_{f_1} + v_{f_2}}{2}.$$
- *Natural:* assume that $f_1(x), \ldots, f_k(x)$ are *independent*: none of the v_{f_i}'s can reconstructed from the others.

6. Two Main Ways of Helping the Poor

- One possibility is to allocate a certain fixed amount of money (or goods) a to each poor person.
- *Example:* US food stamps.
- In this case, the original incomes change from x_i to $x'_i = x_i + a$.
- Another possibility is to provide tax deductions to all the poor people.
- *Example:* tax deductions in the US.
- Since taxes are usually proportional to the income x_i, income increases to $x'_i = \lambda \cdot x_i$, for some $\lambda > 1$.
- An efficient set of poverty measures should enable us to predict how these measures change when we help.

7. Main Definition

An independent set of poverty measures $f_1(x), \ldots, f_k(x)$ is called *efficient* if the following two properties hold:

- once we know all k poverty values $v_j = \sum_{i=1}^{H} f_j(x_i)$ and $a > 0$, we can uniquely predict the new poverty values

$$v'_j = \sum_{i=1}^{H} f_j(x_i + a);$$

- once we know all k poverty values $v_j = \sum_{i=1}^{H} f_j(x_i)$ and $\lambda > 1$, we can uniquely predict new poverty values

$$v'_j = \sum_{i=1}^{H} f_j(\lambda \cdot x_i).$$

8. Discussion and Main Result

- *Lemma:* The set of FGT measures $f_0(x) = 1$, $f_1(x) = 1 - \dfrac{x}{z}$, and $f_2(x) = \left(1 - \dfrac{x}{z}\right)^2$ is efficient.
- We say that two independent sets of poverty measures $f_1(x), \ldots, f_k(x)$ and $g_1(x), \ldots, g_l(x)$ are *equivalent* if:
  - each $f_j(x)$ depends on $g_1(x), \ldots, g_l(x)$; and
  - each $g_j(x)$ depends on $f_1(x), \ldots, f_k(x)$.
- *Proposition:* The set of FGT measures is equivalent to $\{1, x, x^2\}$.
- *Theorem:* Every efficient independent set of poverty measures $f_1(x), \ldots, f_k(x)$ is equiv. to $\{1, x, x^2, \ldots, x^{k-1}\}$.
- *Corollary:* Every efficient independent set of poverty measures $f_1(x), f_2(x), f_3(x)$ is equiv. to the FGT set.
- *Conclusion:* We have thus justified FGT measures.

9. Conclusions

- Several semi-heuristic poverty measures have been proposed, e.g., Foster-Greer-Thorbecke (FGT) measures.
- FGT measures have worked well on many situations to which they have been applied; however:
  - to be sure that these poverty measures will work in other situations as well,
  - it is desirable to supplement the empirical confirmation with a theoretical justification.
- In this talk:
  - we first use fuzzy logic to provide a *commonsense* interpretation of the FGT measures, and
  - then we transform this commonsense explanation into a *theoretical* justification for these measures.
- This makes us more confident in using FGT measures.

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11. Proof

- Efficiency $\Rightarrow$ if we change the x_i without changing $v_j = \sum\limits_{i=1}^{H} f_j(x_i)$, then $v'_j = \sum\limits_{i=1}^{H} f_j(x_i + a)$ is also unchanged.
- For small Δx_i, the changes are proportional to $f'(x_i)$; so:
  - if $\sum f'_j(x_i) \cdot \Delta x_i = 0$ for all j,
  - then $\sum f'_j(x_i + a) \cdot \Delta x_i = 0$ for all j.
- This requirement can be described in the vector form:
  - if $f'_j \perp \Delta x$ (i.e., $\langle f'_j, \Delta x\rangle = 0$) for all j,
  - then $f'_{aj} \perp \Delta x$ for all j.
- We can thus prove that each f'_{aj} is a linear combination of f'_j:

$$f'_j(x_i + a) = c_{j1}(a) \cdot f'_1(x_i) + \ldots + c_{jk}(a) \cdot f'_k(x_i).$$

12. Proof (cont-d)

- *Reminder:* $f'_j(x_i+a) = c_{j1}(a)\cdot f'_1(x_i)+\ldots+c_{jk}(a)\cdot f'_k(x_i)$.
- So, for $D_j(x) \stackrel{\text{def}}{=} f'_j(x)$, we get
$$D_j(x+a) = c_{j1}(a)\cdot D_1(x)+\ldots+c_{jk}(a)\cdot D_k(x).$$
- Differentiating both sides by a and taking $a=0$, we get $D'_j(x) = c_{j1}\cdot D_1(x)+\ldots+c_{jk}\cdot D_k(x)$.
- Thus, the functions $D_1(x),\ldots,D_k(x)$ satisfy a system of linear differential equations with constant coefficients.
- A general solution to such systems is known: a linear comb. of $x^d\cdot\exp(\alpha\cdot x)$ w/complex α and $d=0,1,2,\ldots$
- For scaling, we similarly get
$$D_j(\lambda\cdot x) = c_{j1}(\lambda)\cdot D_1(x)+\ldots+c_{jk}(\lambda)\cdot D_k(x).$$
- Differentiating, we get
$$x\cdot D'_j(x) = c_{j1}\cdot D_1(x)+\ldots+c_{jk}\cdot D_k(x).$$

13. Proof: Final Part

- For $X = \ln(x)$ and $E_j(X) = D_j(\exp(X))$, we get
$$E'_j(X) = c_{j1} \cdot E_1(X) + \ldots + c_{jk} \cdot E_k(X).$$
- Thus, $E_j(X)$ is a linear combination of functions
$$X^d \cdot \exp(\beta \cdot X).$$
- This means that $D_j(x) = E_j(\ln(x))$ is a linear comb. of
$$(\ln(x))^d \cdot \exp(\beta \cdot \ln(x)) = x^\beta \cdot (\ln(x))^d.$$
- If a f-n $D_j(x)$ contains an exp. term $x^d \cdot \exp(\alpha \cdot x)$, $\alpha \neq 0$, it cannot be represented in the above form.
- Thus, $D_j(x) = f'_j(x)$ is a linear combination of terms x^d with $d = 0, 1, 2, \ldots$, i.e., a polynomial.
- Hence, $f_j(x)$ are also polynomials.
- Now, shift-invariance enables us to conclude that $\{f_j(x)\}$ are equivalent to $\{1, \ldots, x^{k-1}\}$. Q.E.D.