Towards Decision Making under Interval, Set-Valued, Fuzzy, and Z-Number Uncertainty: A Fair Price Approach

Joe Lorkowski and Vladik Kreinovich
University of Texas at El Paso, El Paso, TX 79968, USA
lorkowski@computer.org, vladik@utep.edu

Rafik Aliev
Azerbaijan State Oil Academy, Baki, Azerbaijan
raliev@asoa.edu.az

1. Need for Decision Making

- In many practical situations:
  - we have several alternatives, and
  - we need to select one of these alternatives.
- *Examples:*
  - a person saving for retirement needs to find the best way to invest money;
  - a company needs to select a location for its new plant;
  - a designer must select one of several possible designs for a new airplane;
  - a medical doctor needs to select a treatment for a patient.

2. Need for Decision Making Under Uncertainty

- Decision making is easier if we know the exact consequences of each alternative selection.
- Often, however:
  - we only have an incomplete information about consequences of different alternative, and
  - we need to select an alternative under this uncertainty.

3. How Decisions Under Uncertainty Are Made Now

- Traditional decision making assumes that:
  - for each alternative a,
  - we know the probability $p_i(a)$ of different outcomes i.
- It can be proven that:
  - preferences of a rational decision maker can be described by *utilities* u_i so that
  - an alternative a is better if its expected utility $\overline{u}(a) \stackrel{\text{def}}{=} \sum_i p_i(a) \cdot u_i$ is larger.

4. Hurwicz Optimism-Pessimism Criterion

- Often, we do not know these probabilities p_i.
- For example, sometimes:
  - we only know the range $[\underline{u}, \overline{u}]$ of possible utility values, but
  - we do not know the probability of different values within this range.
- It has been shown that in this case, we should select an alternative s.t. $\alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u} \to \max$.
- Here, $\alpha_H \in [0, 1]$ described the optimism level of a decision maker:
  - $\alpha_H = 1$ means optimism;
  - $\alpha_H = 0$ means pessimism;
  - $0 < \alpha_H < 1$ combines optimism and pessimism.

5. What If We Have Fuzzy Uncertainty? Z-Number Uncertainty?

- There are many semi-heuristic methods of decision making under fuzzy uncertainty.
- These methods have led to many practical applications.
- However, often, different methods lead to different results.
- R. Aliev proposed a utility-based approach to decision making under fuzzy and Z-number uncertainty.
- However, there still are many practical problems when it is not fully clear how to make a decision.
- In this talk, we provide foundations for the new methodology of decision making under uncertainty.
- This methodology which is based on a natural idea of a *fair price*.

6. Fair Price Approach: An Idea

- When we have a full information about an object, then:
  - we can express our desirability of each possible situation
  - by declaring a price that we are willing to pay to get involved in this situation.
- Once these prices are set, we simply select the alternative for which the participation price is the highest.
- In decision making under uncertainty, it is not easy to come up with a fair price.
- A natural idea is to develop techniques for producing such fair prices.
- These prices can then be used in decision making, to select an appropriate alternative.

7. Case of Interval Uncertainty

- *Ideal case:* we know the exact gain u of selecting an alternative.
- *A more realistic case:* we only know the lower bound $\underline{u}$ and the upper bound $\overline{u}$ on this gain.
- *Comment:* we do not know which values $u \in [\underline{u}, \overline{u}]$ are more probable or less probable.
- This situation is known as *interval uncertainty*.
- We want to assign, to each interval $[\underline{u}, \overline{u}]$, a number $P([\underline{u}, \overline{u}])$ describing the fair price of this interval.
- Since we know that $u \leq \overline{u}$, we have $P([\underline{u}, \overline{u}]) \leq \overline{u}$.
- Since we know that $\underline{u}$, we have $\underline{u} \leq P([\underline{u}, \overline{u}])$.

8. Case of Interval Uncertainty: Monotonicity

- *Case 1:* we keep the lower endpoint $\underline{u}$ intact but increase the upper bound.
- This means that we:
  - keeping all the previous possibilities, but
  - we allow new possibilities, with a higher gain.
- In this case, it is reasonable to require that the corresponding price not decrease:

$$\text{if } \underline{u} = \underline{v} \text{ and } \overline{u} < \overline{v} \text{ then } P([\underline{u}, \overline{u}]) \le P([\underline{v}, \overline{v}]).$$

- *Case 2:* we dismiss some low-gain alternatives.
- This should increase (or at least not decrease) the fair price:

$$\text{if } \underline{u} < \underline{v} \text{ and } \overline{u} = \overline{v} \text{ then } P([\underline{u}, \overline{u}]) \le P([\underline{v}, \overline{v}]).$$

9. Additivity: Idea

- Let us consider the situation when we have two consequent independent decisions.
- We can consider two decision processes separately.
- We can also consider a single decision process in which we select a pair of alternatives:
  - the 1st alternative corr. to the 1st decision, and
  - the 2nd alternative corr. to the 2nd decision.
- If we are willing to pay:
  - the amount u to participate in the first process, and
  - the amount v to participate in the second decision process,
- then we should be willing to pay $u+v$ to participate in both decision processes.

10. Additivity: Case of Interval Uncertainty

- About the gain u from the first alternative, we only know that this (unknown) gain is in $[\underline{u}, \overline{u}]$.
- About the gain v from the second alternative, we only know that this gain belongs to the interval $[\underline{v}, \overline{v}]$.
- The overall gain $u + v$ can thus take any value from the interval

$$[\underline{u}, \overline{u}] + [\underline{v}, \overline{v}] \stackrel{\text{def}}{=} \{u + v : u \in [\underline{u}, \overline{u}], v \in [\underline{v}, \overline{v}]\}.$$

- It is easy to check that

$$[\underline{u}, \overline{u}] + [\underline{v}, \overline{v}] = [\underline{u} + \underline{v}, \overline{u} + \overline{v}].$$

- Thus, the additivity requirement about the fair prices takes the form

$$P([\underline{u} + \underline{v}, \overline{u} + \overline{v}]) = P([\underline{u}, \overline{u}]) + P([\underline{v}, \overline{v}]).$$

11. Fair Price Under Interval Uncertainty

- By a *fair price under interval uncertainty*, we mean a function $P([\underline{u}, \overline{u}])$ for which:
  - $\underline{u} \le P([\underline{u}, \overline{u}]) \le \overline{u}$ for all u (*conservativeness*);
  - if $\underline{u} = \underline{v}$ and $\overline{u} < \overline{v}$, then $P([\underline{u}, \overline{u}]) \le P([\underline{v}, \overline{v}])$ (*monotonicity*);
  - (*additivity*) for all $\underline{u}$, $\overline{u}$, $\underline{v}$, and $\overline{v}$, we have

$$P([\underline{u} + \underline{v}, \overline{u} + \overline{v}]) = P([\underline{u}, \overline{u}]) + P([\underline{v}, \overline{v}]).$$

- *Theorem:* Each fair price under interval uncertainty has the form

$$P([\underline{u}, \overline{u}]) = \alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u} \text{ for some } \alpha_H \in [0, 1].$$

- *Comment:* we thus get a new justification of Hurwicz optimism-pessimism criterion.

12. Proof: Main Ideas

- Due to monotonicity, $P([u,u])=u$.
- Due to monotonicity, $\alpha_H \stackrel{\text{def}}{=} P([0,1]) \in [0,1]$.
- For $[0,1]=[0,1/n]+\ldots+[0,1/n]$ (n times), additivity implies $\alpha_H = n\cdot P([0,1/n])$, so $P([0,1/n])=\alpha_H\cdot(1/n)$.
- For $[0,m/n]=[0,1/n]+\ldots+[0,1/n]$ (m times), additivity implies $P([0,m/n])=\alpha_H\cdot(m/n)$.
- For each real number r, for each n, there is an m s.t. $m/n\le r\le (m+1)/n$.
- Monotonicity implies $\alpha_H\cdot(m/n)=P([0,m/n])\le P([0,r])\le P([0,(m+1)/n])=\alpha_H\cdot((m+1)/n)$.
- When $n\to\infty$, $\alpha_H\cdot(m/n)\to\alpha_H\cdot r$ and $\alpha_H\cdot((m+1)/n)\to r$, hence $P([0,r])=\alpha_H\cdot r$.
- For $[\underline{u},\overline{u}]=[\underline{u},\underline{u}]+[0,\overline{u}-\underline{u}]$, additivity implies $P([\underline{u},\overline{u}])=\underline{u}+\alpha_H\cdot(\overline{u}-\underline{u})$. Q.E.D.

13. Case of Set-Valued Uncertainty

- In some cases:
  - in addition to knowing that the actual gain belongs to the interval $[\underline{u}, \overline{u}]$,
  - we also know that some values from this interval cannot be possible values of this gain.
- For example:
  - if we buy an obscure lottery ticket for a simple prize-or-no-prize lottery from a remote country,
  - we either get the prize or lose the money.
- In this case, the set of possible values of the gain consists of two values.
- Instead of a (bounded) *interval* of possible values, we can consider a general bounded *set* of possible values.

14. Fair Price Under Set-Valued Uncertainty

- We want a function P that assigns, to every bounded closed set S, a real number $P(S)$, for which:
  - $P([\underline{u},\overline{u}]) = \alpha_H \cdot \overline{u} + (1-\alpha_H)\cdot \underline{u}$ (*conservativeness*);
  - $P(S+S') = P(S)+P(S')$, where $S+S' \stackrel{\rm def}{=} \{s+s' : s\in S, s'\in S'\}$ (*additivity*).
- *Theorem:* Each fair price under set uncertainty has the form $P(S) = \alpha_H \cdot \sup S + (1-\alpha_H)\cdot \inf S$.
- *Proof: idea.*
  - $\{\underline{s},\overline{s}\} \subseteq S \subseteq [\underline{s},\overline{s}]$, where $\underline{s} \stackrel{\rm def}{=} \inf S$ and $\underline{s} \stackrel{\rm def}{=} \sup S$;
  - thus, $[2\underline{s},2\overline{s}] = \{\underline{s},\overline{s}\} + [\underline{s},\overline{s}] \subseteq S + [\underline{s},\overline{s}] \subseteq [\underline{s},\overline{s}] + [\underline{s},\overline{s}] = [2\underline{s},2\overline{s}]$;
  - so $S+[\underline{s},\overline{s}] = [2\underline{s},2\overline{s}]$, hence $P(S)+P([\underline{s},\overline{s}]) = P([2\underline{s},2\overline{s}])$, and

$$P(S) = (\alpha_H\cdot(2\overline{s}) + (1-\alpha_H)\cdot(2\underline{s})) - (\alpha_H\cdot\overline{s} + (1-\alpha_H)\cdot\underline{s}).$$

15. Crisp Z-Numbers, Z-Intervals, and Z-Sets

- Until now, we assumed that we are 100% certain that the actual gain is contained in the given interval or set.
- In reality, mistakes are possible.
- Usually, we are only certain that u belongs to the interval or set with some probability $p \in (0, 1)$.
- A pair of information and a degree of certainty about this this info is what L. Zadeh calls a *Z-number*.
- We will call a pair (u, p) consisting of a (crisp) number and a (crisp) probability a *crisp Z-number*.
- We will call a pair $([\underline{u}, \overline{u}], p)$ consisting of an interval and a probability a *Z-interval*.
- We will call a pair (S, p) consisting of a set and a probability a *Z-set*.

16. Additivity for Z-Numbers

- *Situation:*
  - for the first decision, our degree of confidence in the gain estimate u is described by some probability p;
  - for the 2nd decision, our degree of confidence in the gain estimate v is described by some probability q.
- The estimate $u+v$ is valid only if both gain estimates are correct.
- Since these estimates are independent, the probability that they are both correct is equal to $p \cdot q$.
- Thus, for crisp Z-numbers (u, p) and (v, q), the sum is equal to $(u+v, p \cdot q)$.
- Similarly, for Z-intervals $([\underline{u}, \overline{u}], p)$ and $([\underline{v}, \overline{v}], q)$, the sum is equal to $([\underline{u}+\underline{v}, \overline{u}+\overline{v}], p \cdot q)$.
- For Z-sets, $(S, p)+(S', q)=(S+S', p \cdot q)$.

17. Fair Price for Z-Numbers and Z-Sets

- We want a function P that assigns, to every crisp Z-number (u, p), a real number $P(u, p)$, for which:
  - $P(u, 1) = u$ for all u (*conservativeness*);
  - for all u, v, p, and q, we have $P(u + v, p \cdot q) = P(u, p) + P(v, q)$ (*additivity*);
  - the function $P(u, p)$ is continuous in p (*continuity*).
- *Theorem:* Fair price under crisp Z-number uncertainty has the form $P(u, p) = u - k \cdot \ln(p)$ for some k.
- *Theorem:* For Z-intervals and Z-sets,

$$P(S, p) = \alpha_H \cdot \sup S + (1 - \alpha_H) \cdot \inf S - k \cdot \ln(p).$$

- *Proof:* $(u, p) = (u, 1) + (0, p)$; for continuous $f(p) \stackrel{\text{def}}{=} (0, p)$, additivity means $f(p \cdot q) = f(p) + f(q)$, so

$$f(p) = -k \cdot \ln(p).$$

18. Case When Probabilities Are Known With Interval Or Set-Valued Uncertainty

- We often do not know the exact probability p.
- Instead, we may only know the interval $\left[\underline{p}, \overline{p}\right]$ of possible values of p.
- More generally, we know the set $\mathcal{P}$ of possible values of p.
- If we only know that $p \in [\underline{p}, \overline{p}]$ and $q \in [\underline{q}, \overline{q}]$, then possible values of $p \cdot q$ form the interval

$$\left[\underline{p} \cdot \underline{q}, \overline{p} \cdot \overline{q}\right].$$

- For sets $\mathcal{P}$ and $\mathcal{Q}$, the set of possible values $p \cdot q$ is the set

$$\mathcal{P} \cdot \mathcal{Q} \stackrel{\text{def}}{=} \{p \cdot q : p \in \mathcal{P} \text{ and } q \in \mathcal{Q}\}.$$

19. Fair Price When Probabilities Are Known With Interval Uncertainty

- We want a function P that assigns, to every Z-number $(u, [\underline{p}, \overline{p}])$, a real number $P(u, [\underline{p}, \overline{p}])$, so that:
  - $P(u, [p, p]) = u - k \cdot \ln(p)$ (*conservativeness*);
  - $P(u + v, [\underline{p} \cdot \underline{q}, \overline{p} \cdot \overline{q}]) = P(u, [\underline{p}, \overline{p}]) + P(v, [\underline{q}, \overline{q}])$ (*additivity*);
  - $P(u, [\underline{p}, \overline{p}])$ is continuous in $\underline{p}$ and $\overline{p}$ (*continuity*).
- *Theorem:* Fair price has the form

$$P(u, [\underline{p}, \overline{p}]) = u - (k - \beta) \cdot \ln(\overline{p}) - \beta \cdot \ln(\underline{p}) \text{ for some } \beta \in [0, 1].$$

- For set-valued probabilities, we similarly have $P(u, \mathcal{P}) = u - (k - \beta) \cdot \ln(\sup \mathcal{P}) - \beta \cdot \ln(\inf \mathcal{P})$.
- For Z-sets and Z-intervals, we have $P(S, \mathcal{P}) =$

$$\alpha_H \cdot \sup S + (1 - \alpha_H) \cdot \inf S - (k - \beta) \cdot \ln(\sup \mathcal{P}) - \beta \cdot \ln(\inf \mathcal{P}).$$

20. Proof

- By additivity, $P(S,\mathcal{P}) = P(S,1) + P(0,\mathcal{P})$, so it is sufficient to find $P(0,\mathcal{P})$.
- For intervals, $P(0,[\underline{p},\overline{p}]) = P(0,\overline{p}) + P(0,[p,1])$, for $p \stackrel{\text{def}}{=} \underline{p}/\overline{p}$.
- For $f(p) \stackrel{\text{def}}{=} P(0,[p,1])$, additivity means
$$f(p\cdot q) = f(p)\cdot f(q).$$
- Thus, $f(p) = -\beta\cdot \ln(p)$ for some β.
- Hence, $P(0,[\underline{p},\overline{p}]) = -k\cdot\ln(\overline{p}) - \beta\cdot\ln(p)$.
- Since $\ln(p) = \ln(\overline{p}) - \ln(\underline{p})$, we get the desired formula.
- For sets $\mathcal{P}$, with $\underline{p} \stackrel{\text{def}}{=} \inf\mathcal{P}$ and $\overline{p} \stackrel{\text{def}}{=} \sup\mathcal{P}$, we have $\mathcal{P}\cdot[\underline{p},\overline{p}] = [\underline{p}^2,\overline{p}^2]$, so $P(0,\mathcal{P})+P(0,[\underline{p},\overline{p}]) = P(0,[\underline{p}^2,\overline{p}^2])$.
- Thus, from known formulas for intervals $[\underline{p},\overline{p}]$, we get formulas for sets $\mathcal{P}$.

21. Case of Fuzzy Numbers

- An expert is often imprecise ("fuzzy") about the possible values.
- For example, an expert may say that the gain is small.
- To describe such information, L. Zadeh introduced the notion of *fuzzy numbers.*
- For fuzzy numbers, different values u are possible with different degrees $\mu(u) \in [0, 1]$.
- The value w is a possible value of $u + v$ if:
  - for some values u and v for which $u + v = w$,
  - u is a possible value of 1st gain, and
  - v is a possible value of 2nd gain.
- If we interpret "and" as min and "or" ("for some") as max, we get *Zadeh's extension principle:*

$$\mu(w) = \max_{u,v:\, u+v=w} \min(\mu_1(u), \mu_2(v)).$$

22. Case of Fuzzy Numbers (cont-d)

- *Reminder:* $\mu(w) = \max\limits_{u,v:\, u+v=w} \min(\mu_1(u), \mu_2(v))$.
- This operation is easiest to describe in terms of α-cuts

$$\mathbf{u}(\alpha) = [u^-(\alpha), u^+(\alpha)] \stackrel{\text{def}}{=} \{u : \mu(u) \geq \alpha\}.$$

- Namely, $\mathbf{w}(\alpha) = \mathbf{u}(\alpha) + \mathbf{v}(\alpha)$, i.e.,

$$w^-(\alpha) = u^-(\alpha) + v^-(\alpha) \text{ and } w^+(\alpha) = u^+(\alpha) + v^+(\alpha).$$

- For product (of probabilities), we similarly get

$$\mu(w) = \max_{u,v:\, u\cdot v=w} \min(\mu_1(u), \mu_2(v)).$$

- In terms of α-cuts, we have $\mathbf{w}(\alpha) = \mathbf{u}(\alpha) \cdot \mathbf{v}(\alpha)$, i.e.,

$$w^-(\alpha) = u^-(\alpha) \cdot v^-(\alpha) \text{ and } w^+(\alpha) = u^+(\alpha) \cdot v^+(\alpha).$$

23. Fair Price Under Fuzzy Uncertainty

- We want to assign, to every fuzzy number s, a real number $P(s)$, so that:
  - if a fuzzy number s is located between $\underline{u}$ and $\overline{u}$, then $\underline{u} \le P(s) \le \overline{u}$ (*conservativeness*);
  - $P(u+v) = P(u) + P(v)$ (*additivity*);
  - if for all α, $s^-(\alpha) \le t^-(\alpha)$ and $s^+(\alpha) \le t^+(\alpha)$, then we have $P(s) \le P(t)$ (*monotonicity*);
  - if μ_n uniformly converges to μ, then $P(\mu_n) \to P(\mu)$ (*continuity*).
- *Theorem.* The fair price is equal to

$$P(s) = s_0 + \int_0^1 k^-(\alpha)\, ds^-(\alpha) - \int_0^1 k^+(\alpha)\, ds^+(\alpha) \text{ for some } k^{\pm}(\alpha).$$

24. Discussion

- $\int f(x)\cdot dg(x) = \int f(x)\cdot g'(x)\,dx$ for a *generalized function* $g'(x)$, hence for generalized $K^{\pm}(\alpha)$, we have:

$$P(s) = \int_0^1 K^-(\alpha)\cdot s^-(\alpha)\,d\alpha + \int_0^1 K^+(\alpha)\cdot s^+(\alpha)\,d\alpha.$$

- Conservativeness means that

$$\int_0^1 K^-(\alpha)\,d\alpha + \int_0^1 K^+(\alpha)\,d\alpha = 1.$$

- For the interval $[\underline{u},\overline{u}]$, we get

$$P(s) = \left(\int_0^1 K^-(\alpha)\,d\alpha\right)\cdot\underline{u} + \left(\int_0^1 K^+(\alpha)\,d\alpha\right)\cdot\overline{u}.$$

- Thus, Hurwicz optimism-pessimism coefficient α_H is equal to $\int_0^1 K^+(\alpha)\,d\alpha$.

- In this sense, the above formula is a generalization of Hurwicz's formula to the fuzzy case.

25. Proof

- Define $\mu_{\gamma,u}(0) = 1$, $\mu_{\gamma,u}(x) = \gamma$ for $x \in (0, u]$, and $\mu_{\gamma,u}(x) = 0$ for all other x.
- $\mathbf{s}_{\gamma,u}(\alpha) = [0, 0]$ for $\alpha > \gamma$, $\mathbf{s}_{\gamma,u}(\alpha) = [0, u]$ for $\alpha \le \gamma$.
- Based on the α-cuts, one check that $s_{\gamma,u+v} = s_{\gamma,u} + s_{\gamma,v}$.
- Thus, due to additivity, $P(s_{\gamma,u+v}) = P(s_{\gamma,u}) + P(s_{\gamma,v})$.
- Due to monotonicity, $P(s_{\gamma,u}) \uparrow$ when $u \uparrow$.
- Thus, $P(s_{\gamma,u}) = k^+(\gamma) \cdot u$ for some value $k^+(\gamma)$.
- Let us now consider a fuzzy number s s.t. $\mu(x) = 0$ for $x < 0$, $\mu(0) = 1$, then $\mu(x)$ continuously $\downarrow 0$.
- For each sequence of values $\alpha_0 = 1 < \alpha_1 < \alpha_2 < \ldots < \alpha_{n-1} < \alpha_n = 1$, we can form an approximation s_n:
  - $s_n^-(\alpha) = 0$ for all α; and
  - when $\alpha \in [\alpha_i, \alpha_{i+1})$, then $s_n^+(\alpha) = s^+(\alpha_i)$.

26. Proof (cont-d)

- Here, $s_n = s_{\alpha_{n-1},s^+(\alpha_{n-1})} + s_{\alpha_{n-2},s^+(\alpha_{n-2})-s^+(\alpha_{n-1})} + \ldots + s_{\alpha_1,\alpha_1-\alpha_2}$.
- Due to additivity, $P(s_n) = k^+(\alpha_{n-1}) \cdot s^+(\alpha_{n-1}) + k^+(\alpha_{n-2}) \cdot (s^+(\alpha_{n-2}) - s^+(\alpha_{n-1})) + \ldots + k^+(\alpha_1) \cdot (\alpha_1 - \alpha_2)$.
- This is minus the integral sum for $\int_0^1 k^+(\gamma)\, ds^+(\gamma)$.
- Here, $s_n \to s$, so $P(s) = \lim P(s_n) = \int_0^1 k^+(\gamma)\, ds^+(\gamma)$.
- Similarly, for fuzzy numbers s with $\mu(x) = 0$ for $x > 0$, we have $P(s) = \int_0^1 k^-(\gamma)\, ds^-(\gamma)$ for some $k^-(\gamma)$.
- A general fuzzy number g, with α-cuts $[g^-(\alpha), g^+(\alpha)]$ and a point g_0 at which $\mu(g_0) = 1$, is the sum of g_0,
  - a fuzzy number with α-cuts $[0, g^+(\alpha) - g_0]$, and
  - a fuzzy number with α-cuts $[g_0 - g^-(\alpha), 0]$.
- Additivity completes the proof.

27. Case of General Z-Number Uncertainty

- In this case, we have two fuzzy numbers:
  - a fuzzy number s which describes the values, and
  - a fuzzy number p which describes our degree of confidence in the piece of information described by s.
- We want to assign, to every pair (s, p) s.t. p is located on $[p_0, 1]$ for some $p_0 > 0$, a number $P(s, p)$ so that:
  - $P(s, 1)$ is as before (*conservativeness*);
  - $P(u + v, p \cdot q) = P(u, p) + P(v, q)$ (*additivity*);
  - if $s_n \to s$ and $p_n \to p$, then $P(s_n, p_n) \to P(s, p)$ (*continuity*).
- *Thm* : $P(s, p) = \int_0^1 K^-(\alpha) \cdot s^-(\alpha)\, d\alpha + \int_0^1 K^+(\alpha) \cdot s^+(\alpha)\, d\alpha +$

$$\int_0^1 L^-(\alpha) \cdot \ln(p^-(\alpha))\, d\alpha + \int_0^1 L^+(\alpha) \cdot \ln(p^+(\alpha))\, d\alpha.$$

28. Conclusions and Future Work

- In many practical situations:
  - we need to select an alternative, but
  - we do not know the exact consequences of each possible selection.
- We may also know, e.g., that the gain will be *somewhat larger* than a certain value u_0.
- We propose to make decisions by comparing the *fair price* corresponding to each uncertainty.
- *Future work:*
  - apply to practical decision problems;
  - generalize to type-2 fuzzy sets;
  - generalize to the case when we have several pieces of information (s, p).

29. Acknowledgments

This work was supported in part:

- by the National Science Foundation grants:
  - HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and
  - DUE-0926721,
- by Grants 1 T36 GM078000-01 and 1R43TR000173-01 from the National Institutes of Health, and
- by grant N62909-12-1-7039 from the Office of Naval Research.