"And"- and "Or"-Operations for "Double", "Triple", etc. Fuzzy Sets

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Traditional Fuzzy . . .
Need for "And'- and . . .
Need to Go Beyond . . .
A Natural Idea
We Need to Extend . . .
"And"-Operations in . . .
"And"-Operations for . . .
MaxEnt Approach . . .

1. Outline

- In the traditional fuzzy logic:
  - the expert's degree of confidence $d(A\,\&\,B)$ in a complex statement $A\,\&\,B$
  - is uniquely determined by his/her degrees of confidence $d(A)$ and $d(B)$ in the statements A and B.
- In practice, for the same degrees $d(A)$ and $d(B)$, we may have different degrees $d(A\,\&\,B)$.
- The best way to take this relation into account is to explicitly elicit the corresponding degrees $d(A\,\&\,B)$.
- If we only elicit information about pairs of statements, then we still need to estimate, e.g., the degree $d(A\,\&\,B\,\&\,C)$.
- In this talk, we explain how to produce such "and"-operations for "double" fuzzy sets.

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2. Traditional Fuzzy Techniques: A Brief Reminder

- Experts often describe their knowledge by using imprecise ("fuzzy") words like "small" or "fast".
- We need to describe this knowledge in computer understandable terms.
- A natural idea is to assign degrees of certainty $d(S) \in [0, 1]$ to expert statements S.
- We can ask an expert to mark his/her degree of certainty by a mark m on a scale from 0 to n, and take

$$d(S) = m/n.$$

- We can also poll n experts; if m of them think that S is true, we take $d(S) = m/n$.

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3. Need for "And'- and "Or"-Operations

- We use expert knowledge to answer queries.
- The answer to a query Q usually depends on several statements.
- What is $d(Q)$?
- For example, Q holds if either S_1 and S_2 hold, or if S_3, S_3, and S_5 hold.
- Thus, to estimate $d(Q)$, we must estimate the degree of certainty in propositional combinations like

$$(S_1 \,\&\, S_2) \vee (S_3 \,\&\, S_4 \,\&\, S_5).$$

- Ideally, we should ask the expert's opinion about all such combinations.
- However, for n statements, we have 2^n such combinations, so we cannot ask about all of them.

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4. Need for "And"- and "Or"-Operations (cont-d)

- We cannot ask the expert about degree of certainty in all possible propositional combinations.
- It is therefore necessary to estimate $d(A\,\&\,B)$ based on $d(A)$ and $d(B)$.
- The estimate $f_\&(a,b)$ for $d(A\,\&\,B)$ based on $a = d(A)$ and $b = d(B)$ is known as an *"and"-operation* (*t-norm*).
- Similarly, we need an *"or"-operation* $f_\vee(a,b)$ and a *negation operation* $f_\neg(a)$.
- The most widely used operations are:

$$f_\&(a,b) = \min(a,b), \quad f_\&(a,b) = a \cdot b,$$

$$f_\vee(a,b) = \max(a,b), \quad f_\vee(a,b) = a + b - a \cdot b,$$

$$f_\neg(a) = 1 - a.$$

5. Need to Go Beyond Traditional Fuzzy

- In the traditional fuzzy techniques, we base our estimate of $d(A\,\&\,B)$ only on $d(A)$ and $d(B)$.
- In reality, for the same degrees of belief in A and B, we may have different degrees of belief in $A\,\&\,B$.
- *Example 1:* if $d(A)=0.5$, then $d(\neg A)=1-0.5=0.5$.
- For $B=A$, $d(A)=d(B)=0.5$ and $d(A\,\&\,B)=d(A)=0.5$.
- For $B=\neg A$, $d(A)=d(B)=0.5$ and $d(A\,\&\,B)=0$.
- *Example 2*: $d(\text{50-year-old is old})=0.1,$

$$d(\text{60-year-old is old})=0.8,\ \text{so}$$

$$d_0 \stackrel{\text{def}}{=} d(\text{50-year-old is old}\,\&\,\text{60-year-old is not old}) =$$

$$f_\&(0.1, 1-0.2) > 0 \text{ for } \min(a,b) \text{ and } a\cdot b.$$

- However, intuitively, $d_0=0$.

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6. A Natural Idea

- A natural solution to the above problem is to explicitly elicit and store:
  - not only the expert's degree of confidence $\mu_P(x)$ that a given value x satisfies the property x
  - but also the degree of confidence $\mu_{PP}(x, x')$ that both x and x' satisfy the property P.
- In this approach, to describe a property, we need *two* functions:
  - a function $\mu_P : X \to [0, 1]$, and
  - a function $\mu_{PP} : X \times X \to [0, 1]$ for which

$$\mu_{PP}(x, x') = \mu_{PP}(x', x) \text{ and } \mu_{PP}(x, x') \le \mu_P(x).$$

- Since we need two functions, it is natural to call such pairs (μ_P, μ_{PP}) *double fuzzy sets.*
- We can also ask about the triples (x, x', x'') etc.

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7. We Need to Extend "And"- and "Or"-Operations to "Double", "Triple" etc. Fuzzy Sets

- If we explicitly elicit $d(A \,\&\, B)$, we do not need the usual "and"-operation.
- However, we still need to estimate $d(A \,\&\, B \,\&\, C)$ based on the available values:

$$d(A),\quad d(B),\quad d(C),\quad d(A \,\&\, B),\quad d(A \,\&\, C),\quad d(B \,\&\, C).$$

- We will show that:
  - the ideas behind the most popular t-norms and t-conorms
  - can be used describe the desired "and"- and "or"-operations for the "double" fuzzy sets.

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8. "And"-Operations in Traditional Fuzzy Logic: Reminder

- Traditionally, expert's degrees of certainty are also called *subjective probabilities*.
- In probabilistic terms:
  - we know the probabilities $p(s_1)$ and $p(s_2)$ of two statements s_1 and s_2;
  - we want to estimate the probability $p(s_1 \,\&\, s_2)$.
- Depending on the dependence between s_1 and s_2, we may have different values of $p(s_1 \,\&\, s_2)$.
- There are two main approaches to deal with this non-uniqueness:
  - we can find the range of all possible values $p(s_1 \,\&\, s_2)$;
  - or we can select a single "most probable" value $p(s_1 \,\&\, s_2)$.

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9. Inequalities (Linear Programming) Approach

- We need to know the probabilities of all basic combinations $s_1 \,\&\, s_2$, $s_1 \,\&\, \neg s_2$, $\neg s_1 \,\&\, s_2$, and $\neg s_1 \,\&\, \neg s_2$.
- We know $d_1 = p(s_1)$ and $d_2 = p(s_2)$; based on $x \stackrel{\text{def}}{=} p(s_1 \,\&\, s_2)$, we get:

$$p(s_1 \,\&\, \neg s_2) = p(s_1) - p(s_1 \,\&\, s_2) = d_1 - x,$$

$$p(\neg s_1 \,\&\, s_2) = p(s_2) - p(s_1 \,\&\, s_2) = d_2 - x, \text{ and}$$

$$p(\neg s_1 \,\&\, \neg s_2) = 1 - p(s_1) - p(s_2) + p(s_1 \,\&\, s_2) = 1 - d_1 - d_2 + x.$$

- All the basic probabilities must be non-negative:

$$x \ge 0; \;\; d_1 - x \ge 0; \;\; d_2 - x \ge 0; \;\; 1 - d_1 - d_2 + x \ge 0, \text{ i.e.,}$$

$$x \ge 0; \;\; x \le d_1; \;\; x \le d_2; \;\; x \ge d_1 + d_2 - 1.$$

- So, the range of possible values is

$$\max(d_1 + d_2 - 1, 0) \le x \le \min(d_1, d_2).$$

- Both endpoints serve as possible t-norms.

10. Maximum Entropy (MaxEnt) Approach

- Often, we do not know the exact probabilities.
- It is reasonable not to hide uncertainty, i.e., select a distribution with the largest uncertainty.
- There are reasonable arguments that uncertainty of a probability distribution is best described by its entropy

$$S = -\sum p_i \cdot \ln(p_i).$$

- Here, $p_i = x$, $d_1 - x$, $d_2 - x$, and $1 - d_1 - d_2 + x$, so

$$S = -x \cdot \ln(x) - (d_1 - x) \cdot \ln(d_1 - x) - (d_2 - x) \cdot \ln(d_2 - x) - (1 - d_1 - d_2 + x) \cdot \ln(1 - d_1 - d_2 + x).$$

- Maximizing S results in $x = d_1 \cdot d_2$.
- For "or", inequalities approach leads to

$$\max(a, b) \le x \le \min(a + b, 1).$$

- For "or", MaxEnt leads to $d_1 + d_2 = d_2 \cdot d_2$.

11. "And"-Operations for "Double" Fuzzy Sets

- We know $d_i = p(s_i)$ and $d_{ij} = p(s_i \,\&\, s_j)$, $1 \le i, j \le 3$.
- From $x = p(s_1 \,\&\, s_2 \,\&\, s_3)$, we can describe $d_{\varepsilon_1\varepsilon_2\varepsilon_3} \stackrel{\text{def}}{=} p(s_1^{\varepsilon_1} \,\&\, s_2^{\varepsilon_2} \,\&\, s_3^{\varepsilon_3})$, $\varepsilon_i = \pm$ $(s^+ = s,\ s^- = \neg s)$, as

$$d_{++-} = d_{12} - x, d_{+-+} = d_{13} - x, d_{-++} = d_{23} - x,$$

$$d_{+--} = d_1 - d_{12} - d_{23} + x, d_{-+-} = d_2 - d_{12} - d_{23} + x,$$

$$d_{--+} = d_3 - d_{13} - d_{23} + x, d_{---} = 1 - d_1 - d_2 - d_2 + d_{12} + d_{13} + d_{23} - x.$$

- The requirement that $d_{\varepsilon_1\varepsilon_2\varepsilon_3} \ge 0$ leads to:

$$\max(d_{12} + d_{13} - d_1, d_{12} + d_{23} - d_2, d_{13} + d_{23} - d_3, 0) \le x \le$$

$$\min(d_{12}, d_{13}, d_{23}, 1 - d_1 - d_2 - d_3 + d_{12} + d_{13} + d_{23}).$$

- Both bounds can thus serve as appropriate "and"-operations.
- By using duality $A \vee B = \neg(\neg A \,\&\, \neg B)$, we can get the corresponding "or"-operations.

12. MaxEnt Approach $S = \sum p_i \cdot \ln(p_i) \to \max$

- We get $p_i = x$, $d_{12} - x$, $d_{13} - x$, $d_{23} - x$, $d_1 - d_{12} - d_{13} + x$, $d_2 - d_{12} - d_{23} + x$, $d_3 - d_{12} - d_{23} + x$, and
$$1 - d_1 - d_2 - d_3 + d_{12} + d_{23} + d_{13} - x.$$
- Equation $\dfrac{dS}{dx} = 0$ leads to
$$-\ln(x) + \ln(d_{12} - x) + \ln(d_{13} - x) + \ln(d_{23} - x) +$$
$$\ln(d_1 - d_{12} - d_{13} + x) - \ln(d_2 - d_{12} - d_{23} + x) -$$
$$\ln(d_3 - d_{13} - d_{23} + x) +$$
$$\ln(1 - d_1 - d_2 - d_3 + d_{12} + d_{23} + d_{13} - x) = 0.$$
- If we raise e to the power of both side, we get a 4-th order equation.
- It is actually 3rd order since terms x^4 cancel out.
- By using duality $A \vee B = \neg(\neg A \,\&\, \neg B)$, we can get the corresponding "or"-operations.

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