Approximate Nature of Traditional Fuzzy Methodology Naturally Leads to Complex-Valued Fuzzy Degrees

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1. Outline

- In the traditional fuzzy logic, the experts' degrees of confidence are described by numbers from $[0, 1]$.
- These degree have a clear intuitive meaning.
- Surprisingly, in some applications, it is useful to also consider complex-valued degrees.
- The intuitive meaning of complex-valued degrees is not clear.
- In this talk, we provide a possible explanation for the success of complex-valued degrees.
- We show that these degrees naturally appear due to the approximate nature of fuzzy methodology.
- This explanation makes the use of complex-valued degrees more intuitively understandable.

2. Fuzzy Logic: Reminder

- Experts are not 100% sure about their statements.
- To describe the expert's degree of certainty, fuzzy logic uses numbers from the interval $[0, 1]$.
- For example, if an expert marks his certainty as 8 on a scale of 0 to 10, we take $d = m/n$.
- Ideally, we should elicit expert's degree of confidence in all possible combinations of his/her statements.
- However, there are exponentially many such combinations, so we cannot ask the expert about all of them.
- Thus, we need to estimate $d(A \,\&\, B)$ based on $a = d(A)$ and $b = d(B)$.
- The resulting estimate $f_\&(a, b)$ is known as an "and"-operation (t-norm).

3. How t-Norms Are Determined: Reminder

- We find a t-norm empirically: for several pairs of statements (A_k, B_k),
  - we elicit the degrees $d(A_k)$, $d(B_k)$, and $d(A_k \mathbin{\&} B_k)$,
  - and then we find $f_\&(a, b)$ for which, for all k,

$$d(A_k \mathbin{\&} B_k) \approx f_\&(d(A_k), d(B_k)).$$

- For example, we can use the Least Squares method and find $f_\&(a, b)$ for which

$$\sum_k (d(A_k \mathbin{\&} B_k) - f_\&(d(A_k), d(B_k)))^2 \to \min.$$

- This procedure should lead to real-valued degrees.
- Interestingly, sometimes complex-valued degrees are useful.

4. In Practice, the Situation May Be Somewhat More Complicated

- Sometimes:
  - instead of knowing the expert's degree of belief in the basic statements,
  - we only know the expert's degree of belief in some propositional combinations of the basic statements.
- In this case:
  - first, we need to recover the degrees $d_1, \ldots, d_n$ from the available information;
  - then, we use d_i to estimate the expert's degree of belief in other propositional combinations.
- Ideal case: $d(A \,\&\, B) = f_{\&}(d(A), d(B))$ and $d(A \vee B) = f_{\vee}(d(A), d(B))$.
- In this case, we can recover the desired degrees.

5. Ideal Case: Example

- Example: $f_\&(a,b) = a \cdot b$, $f_\vee(a,b) = a + b - a \cdot b$, and the actual (unknown) values are $d_1 = 0.4$ and $d_2 = 0.6$.
- We only know the values $d(S_1 \,\&\, S_2) = 0.4 \cdot 0.6 = 0.24$ and $d(S_1 \vee S_2) = 0.6 + 0.4 - 0.6 \cdot 0.4 = 0.76$.
- To reconstruct d_i, we form equations $d_1 \cdot d_2 = 0.24$ and $d_1 + d_2 - d_1 \cdot d_2 = 0.76$.
- Adding these equations, we get $d_1 + d_2 = 1$, hence $d_2 = 1 - d_1$.
- Substituting $d_2 = 1 - d_1$ into $d_1 \cdot d_2 = 0.24$, we get $d_1^2 - d_1 + 0.24 = 0$, hence

$$d_1 = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - 0.24} = 0.5 \pm \sqrt{0.25 - 0.24} = 0.5 \pm 0.1.$$

- Thus, $d_1 = 0.4$ or $d_1 = 0.6$, as expected.

6. What Happens When the "And"- and "Or"-Operations Are Only Approximate?

- Let's assume that on average, the expert's reasoning is best described by $f_\&(a, b) = a \cdot b$, $f_\vee(a, b) = a + b - a \cdot b$.
- This does not mean, of course, that we always have $d(A \,\&B) = f_\&(d(A), d(B))$.
- For example, when $S_2 = S_1$ and $d(S_1) = 0.5$, we have $d(S_1 \,\&\, S_2) = d(S_1 \vee S_2) = d(S_1) = 0.5 \neq 0.5 \cdot 0.5$.
- Let us see what happens if we try to reconstruct d_i from $d(S_1 \,\&\, S_2) = 0.5$ and $d(S_1 \vee S_2) = 0.5$.
- From $d_1 \cdot d_2 = 0.5$ and $d_1 + d_2 - d_1 \cdot d_2 = 0.5$, we get $d_2 = 1 - d_1$ and $d_1^2 - d_1 + 0.5 = 0$.
- This equation does not have any real solutions, only complex ones.
- So, it makes sense to use complex degrees d_i.

7. Natural Idea Leads to Complex-Valued Degrees

- From $d_1^2 - d_1 + 0.5 = 0$, we get $d_1 = 0.5 \pm 0.5 \cdot \mathrm{i}$.
- It is difficult to interpret complex-valued degrees.
- So, it is natural, for each such complex-valued degree, to take the closest value from the interval $[0, 1]$.
- For complex numbers, the natural distance is Euclidean distance $d(a_1+a_2\cdot\mathrm{i}, b_1+b_2\cdot\mathrm{i}) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$.
- It is easy to see that for a complex number $a_1 + a_2 \cdot \mathrm{i}$, the closest point on $[0, 1]$ is:
  - the value a_1 is $a_1 \in [0, 1]$;
  - the value 0 is $a_1 < 0$, and
  - the value 1 if $a_1 > 1$.
- Thus, for $0.5 \pm 0.5 \cdot \mathrm{i}$, the closest number from $[0, 1]$ is 0.5: exactly what the expert assigned!

8. A Slightly More General Example

- Let's consider the same "and"- and "or"-operations and $S_1 = S_2$, but with a general $d = d(S_1) = d(S_2)$.
- In this example, we get a system of equations $d_1 \cdot d_2 = d$ and $d_1 + d_2 - d_1 \cdot d_2 = d$.
- After adding these two equations, we get $d_1 + d_2 = 2d$, hence $d_2 = 2d - d_1$.
- Substituting $d_2 = 2d - d_1$ into the first equation, we get $d_1 \cdot (2d - d_1) = d$ and $d_1^2 - 2d \cdot d_1 + d = 0$.
- Thus, $d_1 = d \pm \sqrt{d - d^2} \cdot \mathrm{i}$.
- For both complex values d_i, the closest number from the interval $[0, 1]$ is the value d.
- This is also exactly what the experts assigned.

9. Complex Numbers Are Not a Panacea

- One may get a false impression that complex numbers always lead to perfect results.
- To avoid this impression, let's consider another example when S_2 implies S_1.
- In this case, $S_1 \,\&\, S_2$ is simply equivalent to S_2, and $S_1 \vee S_2$ is equivalent to S_1.
- So, for example, for $d_1 = 0.6$ and $d_2 = 0.4$, we get $d(S_1 \,\&\, S_2) = 0.4$ and $d(S_1 \vee S_2) = 0.6$.
- In this example, we get a system of equations $d_1 \cdot d_2 = 0.4$ and $d_1 + d_2 - d_1 \cdot d_2 = 0.6$.
- So, $d_1^2 - d_1 + 0.4 = 0$, and $d_1 = 0.5 \pm \sqrt{0.15} \cdot \mathrm{i}$.
- For both d_1, the closest number from $[0, 1]$ is 0.5.
- This is different from 0.4 and 0.6 – though close.

10. Conclusion

- Traditionally, fuzzy logic uses degree from $[0,1]$.
- These degrees have a clear intuitive sense.
- Recently, it turned out that in some practical situations, it is beneficial to use complex-valued degrees.
- The problem is that the intuitive meaning of complex-valued degrees is not clear.
- We showed that an approximate character of "and"- and "or"-operations naturally leads to complex values.
- Specifically, in some situations:
  - we know the expert's degree of belief $d(S_1 \,\&\, S_2)$ and $d(S_1 \vee S_2)$ in $S_1 \,\&\, S_2$ and $S_1 \vee S_2$, and
  - we want to use these degrees to estimate the expert's degrees of belief $d(S_1)$ and $d(S_2)$.

11. Conclusion (cont-d)

- For this, we solve a system of equations $d(S_1 \,\&\, S_2) = f_\&(d(S_1), d(S_2))$ and $d(S_1 \vee S_2) = f_\vee(d(S_1), d(S_2))$.
- In general:
  - the expert's degrees of belief in $S_1 \,\&\, S_2$ and $S_1 \vee S_2$
  - are somewhat different from the estimates obtained by using "and"- and "or"-operations.
- As a result, the corresponding system of equations sometimes does not have solutions from the interval $[0, 1]$.
- The system only has complex-valued solutions.
- On several examples, we show that these complex-valued degree make sense, in the sense that:
  - for each of these estimated degrees $\widetilde{d}(S_i)$,
  - the closest real number from the interval $[0, 1]$ is indeed close to (or even equal to) $d(S_i)$.

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