Membership Functions Representing a Number vs. Representing a Set: Proof of Unique Reconstruction

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1. Outline

- In some cases, a membership function $\mu(x)$ represents an unknown number.
- In many other cases, it represents an unknown crisp set.
- In this case, for each crisp set S, we can estimate the degree $\mu(S)$ to which this set S is the desired one.
- A natural question is:
  - once we know the values $\mu(S)$ corresponding to all possible crisp sets S,
  - can we reconstruct the original membership function?
- We show that the membership function $\mu(x)$ can indeed be uniquely reconstructed from the values $\mu(S)$.

2. Representing a Number vs. Representing a Set

- In some cases, a fuzzy set is used to represent a *number*.
- Example: we ask a person how old is Mary, and this person replies that Mary is young.
- There is an actual number representing age, but we do not know this number.
- Instead, we have a membership function $\mu(x)$ that describes our uncertain knowledge about this value.
- $\mu(x)$ is our degree of confidence that the x has the desired property – e.g., that a person of age x is young.
- In other cases, a fuzzy set is used to represent not a single crisp *value*, but rather a whole crisp *set*.

3. Representing a Number vs. Representing a Set (cont-d)

- Example of a fuzzy number representing a set:
  - when designing a control system for an autonomous car,
  - we can ask a driver which velocities are safe on a certain road segment.
- In reality, there is a (crisp) set of such values. However, we do not know this set.
- Instead, we have a fuzzy set that describes our uncertain knowledge about this unknown set.
- Here, $\mu(x)$ is our degree of confidence that this element x belongs to the (unknown) set U.
- Thus, our degree of confidence that the element y does not belong to the actual set U is equal to $1 - \mu(y)$.

4. Analysis of the Situation

- Fuzziness means that we do not know the actual set U exactly.
- In other words, several different crisp sets S are possible candidates for the unknown actual set U.
- For each crisp set S, let us estimate our degree of confidence $\mu(S)$ that this set S is the set U.
- The equality $S = U$ means that:
  - for every $x \in S$, we have $x \in U$, and
  - for every $y \notin S$, we have $y \notin U$.
- In other words, if we consider all $x_i \in S$ and all $y_j \notin S$, then $S = U$ means that

$$x_1 \in U \text{ and } x_1 \in U \text{ and } \ldots \text{ and } y_1 \notin U \text{ and } y_2 \notin U \text{ and } \ldots$$

5. Analysis of the Situation (cont-d)

- Our degree of confidence in the above "and"-statement can be obtained by applying an "and"-operation:

$$\mu(S) = f_\&(\mu(x_1), \mu(x_2), \ldots, 1 - \mu(y_1), 1 - \mu(y_2), \ldots).$$

- In fuzzy logic, there are many possible "and"-operations (t-norms).
- However, for most of them (e.g., for $a \cdot b$) the result of applying this operation to infinitely many values is 0.
- Among the most widely used t-norms, the only "and"-operation for which the result is non-0 is min, so

$$\mu(S) = \min\left(\inf_{x \in S} \mu(x), \inf_{y \notin S}(1 - \mu(y))\right).$$

6. Relation to Possibility and Belief

- Similar expressions describe *possibility degree* and *degree of belief*:

$$\text{Poss}(S) = \sup_{x \in S} \mu(x)$$

$$\text{Bel}(S) = 1 - \text{Poss}(-S) = \inf_{x \in S} \mu(x).$$

- One can see that our degree $\rho(S)$ cane described in terms of plausibility and belief, as

$$\mu(S) = \min(\text{Bel}(S), \text{Bel}(-S)) =$$

$$\min(\text{Bel}(S), 1 - \text{Poss}(S)).$$

7. Why Not Use Probabilities: Advantage of a Fuzzy Approach

- At first glance, it may seem that in this situation, we could also use a probabilistic approach.
- In this case, if we denote the probability that $x \in S$ by $p(x)$, then the probability that $y \notin S$ is equal to

$$1 - p(y).$$

- If we make a usual probabilistic assumption that events $x \in S$ corresponding to different x are independent:

$$\text{Prob}(S = U) = \left(\prod_i p(x_i) \right) \cdot \left(\prod_j (1 - p(y_j)) \right).$$

- However, when we have infinitely many values x_i and y_j, this product becomes a meaningless 0.
- Thus, in general, it is not possible to use the probabilistic approach in this situation.

8. A Natural Question

- We have shown how:
  - if we know the original membership function $\mu(x)$,
  - then we can determine the degree $\mu(S)$ for each crisp set S.
- *Natural question*: how uniquely can we reconstruct $\mu(x)$ from $\mu(S)$?
- In other words:
  - if we know the value $\mu(S)$ for every crisp set S,
  - can we uniquely reconstruct the original membership function $\mu(x)$?

9. This Question Is Non-Trivial

- At first glance, it may seem that this reconstruction is easy: e.g., to find $\mu(a)$, why not take $S = \{a\}$?
- However, one can easily see that this simple approach does not work.
- For example, if $\mu(x_0) = 1$, and we want to find $\mu(a)$ for some $a \neq x_0$, then for $x_0 \notin \{a\}$, we have

$$1 - \mu(x_0) = 1 - 1 = 0.$$

- Thus, $\inf\limits_{y \notin \{a\}} (1 - \mu(y)) = 0$, and so, irrespective of what is the actual value of $\mu(a)$:

$$\mu(\{a\}) = \min\left(\inf_{x \in \{a\}} \mu(x), \inf_{y \notin \{a\}} (1 - \mu(y))\right) = 0.$$

- We therefore need more sophisticated techniques for reconstructing $\mu(x)$ from $\mu(S)$.

10. Main Result

- **Proposition 1.**
  - *Let* $\mu(x)$ *and* $\mu'(x)$ *be membership f-ns, and let*

$$\mu(S)=\min\left(\inf_{x\in S}\mu(x),\inf_{y\notin S}(1-\mu(y))\right)\ \textit{and}$$

$$\mu'(S)=\min\left(\inf_{x\in S}\mu'(x),\inf_{y\notin S}(1-\mu'(y))\right).$$

  - *If* $\mu(S)=\mu'(S)$ *for all crisp sets* $S\subseteq X$, *then*

$$\mu(x)=\mu'(x)\ \textit{for all}\ x.$$

- So, the membership f-n $\mu(x)$ can indeed be uniquely reconstructed if we know $\mu(S)$ for all crisp sets S.
- The proof of the main result consists of several lemmas.

11. Lemma 1: Formulation and Proof

- **Lemma 1.** *For every $a \in X$,*

$$\mu(a) < 0.5 \Leftrightarrow \exists S\,(\mu(S \cup \{a\}) < \mu(S - \{a\})).$$

- Let us first prove that if $\mu(a) < 0.5$, then there exists a set S for which $\mu(S \cup \{a\}) < \mu(S - \{a\})$.
- Let us take $S = \{x : \mu(x) \geq 0.5\}$. In this case, $a \notin S$, so $S - \{a\} = S$ and thus, $\mu(S - \{a\}) = \mu(S)$.
- For the selected set S, for all $x \in S$, we have $\mu(x) \geq 0.5$. Thus, $\inf\limits_{x \in S} \mu(x) \geq 0.5$.
- For all values $y \notin S$, we have $\mu(y) < 0.5$ hence $1 - \mu(y) > 0.5$, thus, $\inf\limits_{y \notin S}(1 - \mu(y)) \geq 0.5$, and

$$\mu(S - \{a\}) = \mu(S) = \min\left(\inf_{x \in S} \mu(x), \inf_{y \notin S}(1 - \mu(y))\right) \geq 0.5.$$

12. Proof of Lemma 1 (cont-d)

- On the other hand, for we have $\mu(a) < 0.5$ for $a \in S \cup \{a\}$, thus $\inf\limits_{x \in S \cup \{a\}} \mu(x) \le \mu(a) < 0.5$ and

$$\mu(S \cup \{a\}) = \min\left(\inf_{x \in S \cup \{a\}} \mu(x), \inf_{y \notin S \cup \{a\}} (1 - \mu(y))\right) \le$$

$$\inf_{x \in S \cup \{a\}} \mu(x) < 0.5.$$

- Thus here, $\mu(S \cup \{a\}) < 0.5 \le \mu(S - \{a\})$, so indeed

$$\mu(S \cup \{a\}) < \mu(S - \{a\}).$$

- The existence of such a set S is proven.
- To complete the proof, let us prove that:
  - if there exists S for which $\mu(S \cup \{a\}) < \mu(S - \{a\})$,
  - then $\mu(a) < 0.5$.

13. Proof of Lemma 1 (conclusion)

- Indeed, both $\mu(S \cup \{a\})$ and $\mu(S - \{a\})$ are minima of infinitely many terms.
- Most of these terms are the same, the only difference is the term corresponding to $x = a$:
  - in $\mu(S \cup \{a\})$, we have the term $\mu(a)$ corresponding to $a \in S \cup \{a\}$, while
  - in $\mu(S - \{a\})$, we have the term $1 - \mu(a)$ corresponding to $a \notin S - \{a\}$.
- If we had $\mu(a) \geq 0.5$, then we would have $\mu(a) \geq 1 - \mu(a)$, and thus, we would have $\mu(S \cup \{a\}) \geq \mu(S - \{a\})$.
- So, from the fact that $\mu(S \cup \{a\}) < \mu(S - \{a\})$, we conclude that we cannot have $\mu(a) \geq 0.5$.
- Thus, we must have $\mu(a) < 0.5$.
- The lemma is proven.

14. Lemma 2: Formulation and Proof

- **Lemma 2.** *For every* $a \in X$,

$$\mu(a) > 0.5 \Leftrightarrow \exists S\,(\mu(S \cup \{a\}) > \mu(S - \{a\})).$$

- Let us first prove that if $\mu(a) > 0.5$, then there exists a set S for which $\mu(S \cup \{a\}) > \mu(S - \{a\})$.
- Let us take $S = \{x : \mu(x) \geq 0.5\}$.
- Here, $a \in S$, so $S \cup \{a\} = S$ and $\mu(S \cup \{a\}) = \mu(S)$.
- As we have shown in the proof of Lemma 1, for this set S, we have $\mu(S) \geq 0.5$, thus, $\mu(S \cup \{a\}) = \mu(S) \geq 0.5$.
- On the other hand, for the set $S - \{a\}$, we have $1 - \mu(a) < 0.5$ for $a \notin S - \{a\}$, thus

$$\inf_{y \notin S - \{a\}} (1 - \mu(y)) \leq 1 - \mu(a) < 0.5.$$

15. Proof of Lemmas 2 (cont-d)

- So, we have:

$$\mu(S-\{a\}) = \min\left(\inf_{x\in S-\{a\}} \mu(x), \inf_{y\notin S-\{a\}} (1-\mu(y))\right) \le$$

$$\inf_{y\notin S-\{a\}} (1-\mu(y)) < 0.5.$$

- Thus here, $\mu(S-\{a\}) < 0.5 \le \mu(S\cup\{a\})$, so indeed $\mu(S\cup\{a\}) > \mu(S-\{a\})$.
- The existence of such a set S is proven.
- To complete the proof, let us prove that:
  - if there exists S for which $\mu(S\cup\{a\}) > \mu(S-\{a\})$,
  - then $\mu(a) > 0.5$.
- Indeed, both $\mu(S\cup\{a\})$ and $\mu(S-\{a\})$ are minima of infinitely many terms.

16. Proof of Lemmas 2 (conclusion)

- Most of these terms are the same, the only difference is the term corresponding to $x = a$:
  - in $\mu(S \cup \{a\})$, we have the term $\mu(a)$ corresponding to $a \in S \cup \{a\}$, while
  - in $\mu(S - \{a\})$, we have the term $1 - \mu(a)$ corresponding to $a \notin S - \{a\}$.
- If we had $\mu(a) \le 0.5$, then we would have $\mu(a) \le 1 - \mu(a)$, and thus, we would have $\mu(S \cup \{a\}) \le \mu(S - \{a\})$.
- So, from the fact that $\mu(S \cup \{a\}) > \mu(S - \{a\})$, we conclude that we cannot have $\mu(a) \le 0.5$.
- Thus, we must have $\mu(a) > 0.5$.
- The lemma is proven.

17. Discussion

- According to Lemmas 1 and 2:
  - once we know the values $\mu(S)$ for all crisp sets S,
  - we can then, for each element $a \in X$, check whether $\mu(a) < 0.5$ and whether $\mu(a) > 0.5$.
- If for some element $a \in X$, none of these two inequalities is satisfied, then we can conclude that $\mu(a) = 0.5$.
- So, for these elements a, we can indeed reconstruct the value $\mu(a)$.
- Let us show that we can reconstruct $\mu(a)$ also for the elements a for which $\mu(a) < 0.5$ or $\mu(a) > 0.5$.

18. Lemma 3: Formulation and Proof

- **Lemma 3.** *If* $\mu(a) < 0.5$, *then*

$$\mu(a) = \sup_{S:\, a \in S} \mu(S).$$

- To prove Lemma 3, we must prove:
  - that for every set S that contains the element a, we have $\mu(S) \le \mu(a)$, and
  - that there exists a set S that contains the element a and for which $\mu(S) = \mu(a)$.
- Let us first prove that when $a \in S$, then $\mu(S) \le \mu(a)$.
- Indeed, by definition of $\mu(S)$, we have

$$\mu(S) = \min\left(\inf_{x \in S} \mu(x), \inf_{y \notin S} (1 - \mu(y))\right) \le \inf_{x \in S} \mu(x) \le \mu(a).$$

- Let us now prove that there exists a set S that contains the element a and for which $\mu(S) = \mu(a)$.

19. Proof of Lemma 3 (cont-d)

- As such a set, we can take $S = \{x : \mu(x) \ge 0.5\} \cup \{a\}$.
- For this set, for elements $x \in S$ for which $\mu(x) \ge 0.5$, we have $\mu(x) \ge 0.5$.
- For the element $a \in S$, we have $\mu(a) < 0.5$.
- Thus, the smallest of the values $\mu(x)$ for all $x \in S$ is the value $\mu(a)$: $\inf\limits_{x \in S} \mu(x) = \mu(a)$.
- For elements $y \notin S$, we have $\mu(y) < 0.5$, thus $1 - \mu(y) > 0.5$ and hence,

$$\inf_{y \notin S}(1 - \mu(y)) \ge 0.5 > \mu(a) = \inf_{x \in S} \mu(x).$$

- So, $\mu(S) = \min\left(\inf\limits_{x \in S} \mu(x), \inf\limits_{y \notin S}(1 - \mu(y))\right) = \mu(a)$.
- The lemma is proven.

20. The Reconstruction Formula from Lemma 3 Makes Common Sense

- An element a is possible if there exists a set S containing this element a which is possible.
- From the common sense viewpoint, "there exists" means "or":
  - either one of the sets S containing a is possible,
  - or another one, etc.
- Thus, the degree that a is possible can be obtained by:
  - applying an "or"-operation
  - to statements "S is possible" corresponding to different $S \ni a$.
- There are infinitely many such sets, so we need to select an operation that does not lead to 1, thus max.

21. An Example When This Common Sense Formula Is Not Sufficient

- Let $X = [0,1]$ and $\mu(x) = x$, then $\mu(1) = 1$.
- However, for all sets $S \ni 1$, we have $\mu(S) \le 0.5$.
- Indeed, if $0.5 \in S$, then:

$$\mu(S) = \min\left(\inf_{x\in S}\mu(x), \inf_{y\notin S}(1-\mu(y))\right) \le$$

$$\inf_{x\in S}\mu(x) \le \mu(0.5) = 0.5.$$

- If $0.5 \notin S$, then:

$$\mu(S) = \min\left(\inf_{x\in S}\mu(x), \inf_{y\notin S}(1-\mu(y))\right) \le$$

$$\inf_{y\notin S}(1-\mu(y)) \le 1-\mu(0.5) = 1-0.5 = 0.5.$$

- In both cases, $\mu(S) \le 0.5$, thus, $\sup\limits_{S:\,1\in S}\mu(S) \le 0.5$ and thus, $\sup\limits_{S:\,1\in S}\mu(S) < \mu(1) = 1$.

22. Lemma 4: Formulation and Proof

- **Lemma 4.** *If* $\mu(a) > 0.5$, *then*

$$\mu(a) = 1 - \sup_{S:\, a \notin S} \mu(S).$$

- To prove this equality, it is sufficient to prove:
  - that for every $S \not\ni a$, we have $\mu(S) \le 1 - \mu(a)$, and
  - that there exists $S \not\ni a$ for which $\mu(S) = 1 - \mu(a)$.

- Let us first prove that when $a \notin S$, then $\mu(S) \le 1 - \mu(a)$; indeed, by definition of $\mu(S)$, we have

$$\mu(S) = \min\left(\inf_{x \in S} \mu(x), \inf_{y \notin S} (1 - \mu(y))\right) \le$$

$$\inf_{y \notin S} (1 - \mu(y)) \le 1 - \mu(a).$$

- Let us now prove that there exists a set S that does not contain the element a and for which $\mu(S) = 1 - \mu(a)$.

23. Proof of Lemma 4 (cont-d)

- As such a set, we can take $S = \{x : \mu(x) \geq 0.5\} - \{a\}$.
- For this set, for elements $y \notin S$, we have $\mu(y) < 0.5$ and thus, $1 - \mu(y) > 0.5$.
- For the element $a \notin S$, we have $\mu(a) > 0.5$ and thus, $1 - \mu(a) < 0.5$.
- Thus, the smallest of the values $1 - \mu(y)$ for all $y \notin S$ is the value $1 - \mu(a)$: $\inf\limits_{y \notin S}(1 - \mu(y)) = 1 - \mu(a)$.
- For elements $x \in S$, we have $\mu(x) \geq 0.5$, thus

$$\inf_{x \in S} \mu(x) \geq 0.5 > 1 - \mu(a) = \inf_{y \notin S}(1 - \mu(y)).$$

- So, $\mu(S) = \min\left(\inf\limits_{x \in S} \mu(x), \inf\limits_{y \notin S}(1 - \mu(y))\right) = 1 - \mu(a)$.
- The lemma is proven, and so is the proposition.

24. Auxiliary Result: Which Crisp Set Is the Most Probable?

- In principle, we can have many different crisp sets S with different degrees $\mu(S)$.
- A natural question is: which crisp sets S are the most probable ones?
- In other words, which crisp sets have the largest degree $\mu(S)$?
- **Proposition 2.** *For every membership function $\mu(x)$ and crisp set S, the following conditions are equivalent:*
  - *the set S has the largest possible value $\mu(S)$, and*
  - *the set S contains all a with $\mu(a) > 0.5$ and does not contain any a with $\mu(a) < 0.5$.*
- *Comment:* it doesn't matter whether we include a with $\mu(a) = 0.5$ or not, the value $\mu(S)$ will not change.

25. Discussion

- This result is in good accordance with common sense.
- Indeed:
  - the inequality $\mu(a) > 0.5$ is equivalent to $\mu(a) > 1 - \mu(a)$, and
  - the inequality $\mu(a) < 0.5$ is equivalent to $\mu(a) < 1 - \mu(a)$.
- So:
  - If our degree of confidence $\mu(a)$ that $a \in U$ is greater than the degree $1 - \mu(a)$ that $a \notin U$,
  - then we add this element a to the set.
- On the other hand:
  - If our degree of confidence $1 - \mu(a)$ that $a \notin U$ is greater than the degree $\mu(a)$ that $a \in U$,
  - then we do not add this element a to the set.

26. Remaining Questions

- An expert is often unable to describe his/her degree of confidence by a single number.
- In such situations, a reasonable idea is to allow an *interval* of possible values of degree of confidence.
- Interval-valued membership functions $\mu(x) = [\underline{\mu}(x), \overline{\mu}(x)]$ were successful in many applications.
- In the interval case, it is natural to define $1 - [\underline{a}, \overline{a}]$ as the set of all the values $1 - a$ when $a \in [\underline{a}, \overline{a}]$, so:

$$1 - [\underline{a}, \overline{a}] = [1 - \overline{a}, 1 - \underline{a}].$$

- It is natural to define $\min([\underline{a}, \overline{a}], [\underline{b}, \overline{b}])$ as the set of all the values $\min(a, b)$ when $a \in [\underline{a}, \overline{a}]$ and $b \in [\underline{b}, \overline{b}]$, so:

$$\min([\underline{a}, \overline{a}], [\underline{b}, \overline{b}]) = [\min(\underline{a}, \underline{b}), \min(\overline{a}, \overline{b})].$$

27. Remaining Questions (cont-d)

- In the interval-valued case, we can similarly define, for each crisp set S, the interval $\mu(S)$ as follows:

$$\mu(S) = \min\left(\inf_{x \in S} \mu(x), \inf_{y \notin S}(1 - \mu(y))\right).$$

- It is then reasonable to ask a similar question:
  - once we know the intervals $\mu(S)$ corresponding to all possible crisp sets S,
  - can we uniquely reconstruct the original interval-valued membership function $\mu(x)$?
- A similar question can be formulated when we consider type-2 fuzzy sets, when:
  - each value $\mu(x)$ is not necessarily an interval,
  - but can be any fuzzy number.

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