How to Detect Crisp Sets Based on Subsethood Ordering of Normalized Fuzzy Sets? How to Detect Type-1 Sets Based on Subsethood Ordering of Normalized Interval-Valued Fuzzy Sets?

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1. Introduction

- A fuzzy set is usually defined as function A from a certain set U (*Universe of discourse*) to $[0, 1]$.
- Traditional – "crisp" – sets can be viewed as particular cases of fuzzy sets, for which $A(a) \in \{0, 1\}$ for all x.
- In most applications, we consider *normalized* fuzzy sets, i.e., fuzzy sets for which $A(x) = 1$ for some $x \in U$.
- For crisp sets, this corresponds to considering non-empty sets.
- For two crisp sets, A is a subset or B if and only if $A(x) \le B(x)$ for all x.
- The same condition is used as a definition of the subsethood ordering between fuzzy sets:
  - a fuzzy set A is a *subset* of a fuzzy set B
  - if $A(x) \le B(x)$ for all x.

2. Introduction (cont-d)

- Subsets $B \subseteq A$ which are different from the set A are called *proper* subsets of A.
- A natural question is:
  - if we have a class of all normalized fuzzy sets with the subsethood relation,
  - can we detect which of these fuzzy sets are crisp?
- It is known that:
  - if we alow *all* possible fuzzy sets – even non-normalized ones,
  - then we can detect crisp sets.
- In this talk, we show that such a detection is possible even if we restrict ourselves only to normalized sets.

3. Results

- We want to describe general crisp sets in terms of subsethood relation $\subseteq$ between fuzzy sets.
- For this purpose, let us first describe some auxiliary notions in these terms.
- In this part of the talk, we only consider normalized fuzzy sets.
- **Proposition.**
  - *A normalized fuzzy set is a 1-element crisp set*
  - *if and only if it has no proper normalized fuzzy subsets, i.e., if and only if $B \subseteq A$ implies $B = A$.*
- Let us first prove that:
  - a 1-element crisp set $A = \{x_0\}$ (i.e., a set for which $A(x_0) = 1$ and $A(x) = 0$ for all $x \neq x_0$)
  - has the desired property.

4. Proof of the First Auxiliary Result (cont-d)

- Indeed, if $B \subseteq A$, then $B(x) \le A(x)$ for all x.
- For $x \ne x_0$, we have $A(x) = 0$, so we have $B(x) = 0$ as well.
- Since B is a normalized fuzzy set, it has to attain value 1 somewhere.
- We have $B(x) = 0$ for all $x \ne x_0$.
- So, the only point $x \in U$ at which $B(x) = 1$ is the point x_0.
- Thus, we have $B(x_0) = 1$.
- So, indeed, we have $B(x) = A(x)$ for all x, i.e., $B = A$.

5. Proof of the First Auxiliary Result (cont-d)

- Vice versa, let us prove that:
  - each normalized fuzzy set A which is different from a 1-element crisp set
  - has a proper normalized fuzzy subset.
- Indeed, since A is normalized, we have $A(x_0) = 1$ for some x_0.
- Then, we can take $B = \{x_0\}$.
- Clearly, $B \subseteq A$, and, since A is not a 1-element crisp set, $B \neq A$.
- The proposition is proven.

6. Second Auxiliary Result

- **Definition.** *By a* 2-element set, *we mean a normalized fuzzy set* A *for which* $A(x) > 0$ *for exactly two* $x \in U$.
- **Proposition.**
  - *Let* A *be a normalized fuzzy set* A *which is not a 1-element crisp set.*
  - *Then, the following two conditions are equivalent to each other:*
    - A *is a non-crisp 2-element set, and*
    - *the class* $\{B : B \subseteq A\}$ *is linearly ordered, i.e.:*

$$\text{if } B_1, B_2 \subseteq A \text{ then } B_1 \subseteq B_2 \text{ or } B_2 \subseteq B_1.$$

7. Third Auxiliary Result

- **Proposition.** *A normalized fuzzy set A is a crisp 2-element set $\Leftrightarrow$ the following 2 conditions hold:*
  - *the set A itself is not a 1-element crisp set and not a 2-element non-crisp set, but*
  - *each proper norm. fuzzy subset $B \subseteq A$ is either a crisp 1-element sets or a non-crisp 2-element set.*

8. Main Result

- **Proposition.** *A normalized fuzzy set is crisp if and only if we have one of the following two cases:*
  - *A is a 1-element fuzzy set, or*
  - *for every subset $B \subseteq A$ which is a non-crisp 2-element set, $\exists$ a crisp 2-element set C for which*

$$B \subseteq C \subseteq A.$$

- Previous propositions show that the following properties can be described in terms of subsethood:
  - of being a crisp 1-element set,
  - of being a crisp 2-element set, and
  - of being a non-crisp 2-element set.
- Thus, this Proposition shows that crispness can indeed be described in terms of subsethood.

9. Interval-Valued Case

- The traditional fuzzy logic assumes that:
  - experts can meaningfully describe their degrees of certainty
  - by numbers from the interval $[0, 1]$.
- In practice, however, experts cannot meaningfully select a single number describing their certainty.
- Indeed, it is not possible to distinguish between, say, degrees 0.80 and 0.81.
- A more adequate description of the expert's uncertainty is:
  - when we allow to characterize the uncertainty
  - by a whole range of possible numbers, i.e., by an interval $\left[\underline{A}(x), \overline{A}(x)\right]$.

10. Interval-Valued Case (cont-d)

- This idea leads to *interval-valued* fuzzy numbers, i.e., mappings that assign,
  - to each element x from the Universe of discourse,
  - an interval $A(x) = [\underline{A}(x), \overline{A}(x)]$.
- For two interval-valued degrees $A = [\underline{A}, \overline{A}]$ and $B = [\underline{B}, \overline{B}]$, it is reasonable to say that $A \le B$ if

$$\underline{A} \le \underline{B} \text{ and } \overline{A} \le \overline{B}.$$

- Thus, we can define a subsethood relation between two interval-valued fuzzy sets A and B as

$$A(x) \le B(x) \text{ for all } x.$$

- An interval-valued fuzzy set is normalized if $\overline{A}(x_0) = 1$ for some x_0.
- Traditional (*type-1*) fuzzy sets can be viewed as particular cases of interval-valued fuzzy sets.

11. Interval-Valued Case (cont-d)

- Namely, they correspond to "degenerate" intervals

$$[A(x), A(x)].$$

- Here, we have a similar problem:
  - can we detect traditional fuzzy sets
  - based only on the subsethood relation between interval-valued fuzzy sets?
- Let us show that this is indeed possible.

12. Interval-Valued: First Auxiliary Result

- **Definition.** *By an* uncertain 1-element set, *we mean a normalized interval-valued fuzzy set A for which*

  $$\exists x_0 \in U\,(A(x_0) = [0,1]\,\&\,(A(x) = [0,0] \text{ for all other } x)).$$

- **Proposition.** *A normalized interval-valued fuzzy set A:*
  - *is an uncertain 1-element set if and only if*
  - *it has no proper normalized subsets.*
- So, we can determine uncertain 1-element sets based on the subsethood relation.

13. Interval-Valued: Second Auxiliary Result

- **Definition.** *By a* basic 1-element set, *we mean a normalized interval-valued fuzzy set A for which:*

$$\exists x_0 \in U\,((A(x_0) = [a, 1] \text{ for some } a > 0)\,\&$$

$$(A(x) = [0, 0] \text{ for all } x \neq x_0)).$$

- **Definition.** *By a* basic 2-element set, *we mean a norm. interval-valued fuzzy set A s.t. for some $x_0 \neq x_1$:*
  - $A(x_0) = [0, 1]$,
  - $A(x_1) = [0, a]$ *for some* $a \in (0, 1)$, *and*
  - $A(x) = [0, 0]$ *for all other* x.

14. Interval-Valued: 2nd Aux. Result (cont-d)

- **Proposition.**
  - *Let A be a normalized interval-valued fuzzy set which is not an uncertain 1-element set.*
  - *Then, the following two conditions are equivalent to each other:*
    - *the class $\{B : B \subseteq A\}$ of all subsets of A is linearly ordered;*
    - *A is either a basic 1-element set or a basic 2-element set.*
- So, we can determine, based on the subsethood relation, whether A is a basic set.

15. Interval-Valued: Third Auxiliary Result

- **Proposition.** *If A is a basic 1- or 2-element set, then the following properties are equivalent:*
  - *A is a crisp 1-element set;*
  - *no proper superset of A is a basic 1-element set or a basic 2-element set.*
- So, we can determine crisp 1-element sets based only on the subsethood relation.

16. Interval-Valued: Fourth Auxiliary Result

- **Proposition.** *For a normalized interval-valued fuzzy set, the following two conditions are satisfied:*
  - *A is either an uncertain 1-element set or a basic 1-element set;*
  - *A is a subset of a crisp 1-element set.*
- **Proof:** straightforward.
- We know how to describe, based on the subsethood relation:
  - when A is an uncertain 1-element set, and
  - when A is a basic set,
- We can therefore determine basic 1-element sets and basic 2-element sets based on subsethood relation only.

17. Interval-Valued: Fifth Auxiliary Result

- **Definition.**
  - *Let A be a basic 2-element set, with:*
    - $A(x_0) = [0, 1]$,
    - $A(x_1) = [0, a]$ *for some* $a \in (0, 1)$, *and*
    - $A(x) = [0, 0]$ *for all other* x.
  - *Then, by its* type-1 cover, *we mean a normalized interval-valued fuzzy set* A' *for which:*
    - $A'(x_0) = [1, 1]$,
    - $A'(x_1) = [a, a]$, *and*
    - $A'(x) = [0, 0]$ *for all other* x.
- Let us show that the type-1 cover can be determined in terms of the subsethood relation.

18. Fifth Auxiliary Result (cont-d)

- **Proposition.** *Let A be a basic 2-element set. Then:*
  - *its type-1 cover A' is the $\subseteq$-smallest normalized interval-valued fuzzy set*
  - *that contains all the normalized interval-valued sets $B \supseteq A$ for which the following conditions hold:*
    - *the set B is not a basic 2-element set;*
    - *the class of all basic 2-element subsets of B is linearly ordered;*
    - *the class $\{C : C \text{ is normalized } \& \; A \subseteq C \subseteq B\}$ is linearly ordered; and*
    - *the set B has only one uncertain 1-element subset.*

19. Interval-Valued: Main Result

- **Definition.**
  - *Let A be an uncertain 1-element set, with $A(x_0) = [0,1]$, and $A(x) = [0,0]$ for all other x.*
  - *Then, by its* type-1 cover, *we mean a crisp set*

$$A' = \{x_0\}.$$

- **Proposition.** *A normalized interval-valued fuzzy set is a type-1 set $\Leftrightarrow$ the following conditions hold:*
  - *if $B \subseteq A$ for some uncertain 1-element set, then $B' \subset A$, and*
  - *if $B \subseteq A$ for some basic 2-element set, then*

$$B' \subseteq A.$$

20. Interval-Valued: Main Result (cont-d)

- We have shown that following can all be described in terms of the subsethood relation:
  - the operation B',
  - uncertain 1-element sets, and
  - basic 2-element sets.
- We can thus conclude that:
  - we can detect type-1 sets
  - based on the subsethood relation between normalized interval-valued fuzzy sets.

21. First Conclusion

- In this talk, we consider the following situation.
- We are given the class of all possible normalized fuzzy sets A on a given Universe of discourse X.
- We do not know the values $A(x)$ for $x \in X$.
- We do not even know which of these fuzzy sets are actually crisp and which are not.
- The only information we have about these fuzzy sets in which of them are subsets of others.
- Based on this information, can we detect crisp sets?
- The first conclusion of this talk is that yes, such detection *is* possible.

22. Second Conclusion

- Suppose now that:
  - instead of the class of all "usual" (type-1) normalized fuzzy sets,
  - we now have the class of all normalized *interval-valued* fuzzy sets.
- We do not know the values $A(x) = [\underline{A}(x), \overline{A}(x)]$.
- We do not even know which of these interval-valued fuzzy sets are actually regular (type-1) fuzzy sets.
- The only information that we have about these sets in which of them are subsets of others.
- Based on this information, can we detect type-1 fuzzy sets?
- The second conclusion of this talk is that yes, such detection is also possible.

23. Possible Future Work

- The above results assume that we know *exactly*:
  - which pairs (A, B) of given fuzzy sets are subsets of each other ($A \subseteq B$) and
  - which are not ($A \not\subseteq B$).
- Sometimes:
  - while a fuzzy set A is, strictly speaking, not a subset of a fuzzy set B,
  - it is "almost" a subset, in the sense that few elements of A are outside B.
- To capture this intuition, researchers have developed *subsethood measures* $\sigma(A, B)$.

24. Possible Future Work (cont-d)

- For such measures:
  - if a fuzzy set A is a subset of a fuzzy set B, then $\sigma(A, B) = 1$, and
  - if a fuzzy set A is "almost" a subset of a fuzzy set B, then $\sigma(A, B)$ is smaller than 1 but close to 1;
- These measures turned out to be very useful in image processing.
- The first seemingly natural question is then: what if
  - instead of simply knowing which fuzzy set is a subset of which,
  - we know, for each pair (A, B), the degree $\sigma(A, B)$ to which A is a subset of B.
- Can we then detect crisp set?
- The answer to this question is: definitely yes.

25. Possible Future Work (cont-d)

- Indeed:
  - if we know the values $\sigma(A, B)$ for all A and B,
  - then, by checking when $\sigma(A, B) = 1$, we will also know when $A \subseteq B$,
  - and thus, based on our first result, we can detect crisp sets.
- But what if only know the degrees $\sigma(A, B)$ with some uncertainty $\varepsilon > 0$?
- This is a natural assumption, taking into account that in practice, all the values are usually known with some uncertainty.
- In this case, we probably cannot exactly detect which fuzzy sets are crisp.

26. Possible Future Work (cont-d)

- But can we then,
  - based on the imprecisely known subsethood degrees,
  - detect fuzzy sets which are, in some reasonable sense, almost crisp?
- This would be interesting to find out.

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28. Second Auxiliary Result: Reminder

- **Definition.** *By a* 2-element set, *we mean a normalized fuzzy set A for which $A(x) > 0$ for exactly two $x \in U$.*
- **Proposition.**
  - *Let A be a normalized fuzzy set A which is not a 1-element crisp set.*
  - *Then, the following two conditions are equivalent to each other:*
    - *A is a non-crisp 2-element set, and*
    - *the class $\{B : B \subseteq A\}$ is linearly ordered, i.e.:*

$$\text{if } B_1, B_2 \subseteq A \text{ then } B_1 \subseteq B_2 \text{ or } B_2 \subseteq B_1.$$

29. Proof of the Second Auxiliary Result

- Let us first prove that if A is a 2-element non-crisp set, then the class of all its subsets is linearly ordered.
- Indeed, since A is a normalized fuzzy set, we must have $A(x_0) = 1$ for some $x_0 \in U$.
- Since A is a 2-element set, there must be one more value $x \in U$ for which $A(x) > 0$.
- Let us denote this value by x_1. So, we have:
  - $A(x_0) = 1$,
  - $A(x_1) > 0$ and
  - $A(x) = 0$ for all other $x \in U$.
- If we had $A(x_1) = 1$, then A would be a crisp set – namely, we would have $A = \{x_0, x_1\}$.

30. Proof of the Second Auxiliary Result (cont-d)

- Since A is a non-crisp set, we thus cannot have $A(x_1) = 1$, so we have $0 < A(x_1) < 1$.
- If B is a normalized fuzzy set for which $B \subseteq A$, then:
  - for all x different from x_0 and x_1,
  - we have $B(x) \le A(x) = 0$ and thus, $B(x) = 0$.
- Since B is normalized, we have $B(x) = 1$ for some x.
  - This x cannot be different from x_0 and x_1 – because then $B(x) = 0$.
  - This x cannot be equal to x_1, since then we would have $1 = B(x_1) \le A(x_1) < 1$ and $1 < 1$.
- Thus, this x must be equal to x_0, $B(x_0) = 1$.

31. Proof of the Second Auxiliary Result (cont-d)

- So, all fuzzy normalized subsets B of the set A have the following form:
  - $B(x_0) = 1$,
  - $B(x_1) \le A(x_1)$, and
  - $B(x) = 0$ for all other x.
- For two such subsets, we can have:
  - either $B_1(x_1) \le B_2(x_1)$,
  - or $B_2(x_1) \le B_1(x_1)$.
- One can easily check that:
  - if $B_1(x_1) \le B_2(x_1)$, then $B_1(x) \le B_2(x)$ for all x and thus, $B_1 \subseteq B_2$;
  - similarly, if $B_2(x_1) \le B_1(x_1)$, then $B_2(x) \le B_1(x)$ for all x and thus, $B_2 \subseteq B_1$.

32. Proof of the Second Auxiliary Result (cont-d)

- So, for every two normalized fuzzy subsets B_1 and B_2 of the set A, we have either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.
- Thus, the class of all such subsets is indeed linearly ordered.
- To complete the proof, let us now prove that:
  - if a normalized fuzzy set A is not a 1-element fuzzy set and *not* a non-crisp 2-element set,
  - then the class $\{B : B \subseteq A\}$ is *not* linearly ordered,
  - i.e., there exists normalized fuzzy subsets $B_1 \subseteq A$ and $B_2 \subseteq A$ for which $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$.
- The fact that the set A is not a 1-element set means that $A(x) > 0$ for at least two different values x.

33. Proof of the Second Auxiliary Result (cont-d)

- By definition, a non-crisp 2-element set is a normalized fuzzy set:
  - which is a 2-element set *and*
  - which is not crisp.
- So, if a normalized fuzzy set A is *not* a non-crisp 2-element set, this means that it is:
  - either not a 2-element set
  - or it is a crisp 2-element set.
- Let us show that in both cases, we can find subsets $B_1 \subseteq A$ and $B_2 \subseteq A$ for which $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$.

34. Proof of the Second Auxiliary Result (cont-d)

- Let us first consider the case when A is not a 2-element set, i.e., when,
  - in addition to the point x_0 at which $A(x_0) = 1$,
  - there exist at least two other points x_1 and x_2 for which $A(x_1) > 0$ and $A(x_1) > 0$.
- In this case, we can take the following sets B_1 and B_2:
  - $B_1(x_0) = B_2(x_0) = 1$;
  - $B_1(x_1) = A(x_1)$ and $B_2(x_1) = 0$;
  - $B_2(x_1) = 0$ and $B_2(x_2) = A(x_2)$, and
  - $B_1(x) = B_2(x)$ for all other x.
- One can see that $B_1(x) \le A(x)$ and $B_2(x) \le A(x)$ for all x, so indeed $B_1 \subseteq A$ and $B_2 \subseteq A$.

35. Proof of the Second Auxiliary Result (cont-d)

- However, here:
  - $B_1(x_1) = A(x_1) > 0 = B_2(x_1)$, so we cannot have $B_1 \subseteq B_2$, because that would imply $B_1(x_1) \le B_2(x_1)$;
  - similarly, $B_2(x_2) = A(x_2) > 0 = B_1(x_2)$,
  - so we cannot have $B_2 \subseteq B_1$, because that would imply $B_2(x_2) \le B_1(x_2)$.
- So, we indeed have $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$.
- Let us now consider the case when A is a 2-element crisp set, i.e., when $A = \{x_0, x_1\}$.
- In this case, we can take $B_1 = \{x_0\}$ and $B_2 = \{x_1\}$.
- Clearly, $B_1 \subseteq A$, $B_2 \subseteq A$, $B_1 \not\subseteq B_2$, and $B_2 \not\subseteq B_1$.
- So, the proposition is proven.

36. Third Auxiliary Result

- **Proposition.** *A normalized fuzzy set A is a crisp 2-element set* $\Leftrightarrow$ *the following 2 conditions hold:*
  - *the set A itself is not a 1-element crisp set and not a 2-element non-crisp set, but*
  - *each proper norm. fuzzy subset* $B \subseteq A$ *is either a crisp 1-element sets or a non-crisp 2-element set.*
- If A is a 2-element crisp set, i.e., if $A = \{x_0, x_1\}$ for some $x_0 \neq x_1$, then it is clearly:
  - not a 1-element crisp set, and
  - not a non-crisp 2-element set.
- Let us prove that in this case, every proper normalized fuzzy subset $B \subseteq A$ is
  - either a 1-element crisp set
  - or a non-crisp 2-element set.

37. Third Auxiliary Result (cont-d)

- Here, $A(x) > 0$ for only two values $x = x_0$ and $x = x_1$, and $B(x) \le A(x)$ for all x.
- So, the value $B(x)$ can be positive also for at most two values x_i.
- If $B(x) > 0$ for only one value x, then, since B is normalized, for this x, we must have $B(x) = 1$.
- Thus, we have $B = \{x\}$, i.e., B is a 1-element crisp set.
- If $B(x) > 0$ for two different values x, this means that we have $B(x_0) > 0$ and $B(x_1) > 0$.
- Since the set B is normalized, one of these value must be equal to 1.
- If the second one is equal to 1, we will have $B = A$ – but B is a proper subset.

38. Third Auxiliary Result (cont-d)

- Thus, one of the values $B(x_i)$ is smaller than 1 – thus, B is a non-crisp 2-element set.
- Let us now prove that:
  - if a normalized fuzzy set A is not a 2-element crisp set,
  - then one of the above properties is not satisfied.
- In other words, in this case:
  - either A is 1-element crisp set or a 2-element non-crisp set,
  - or one of its proper subsets $B \subseteq A$ is *not* a non-crisp 2-element set.

39. Third Auxiliary Result (cont-d)

- In other words, we want to prove that if A is:
  - not a crisp 1-element set, not a crisp 2-element set, and not a non-crisp 2-element set,
  - then one of its proper subsets $B \subseteq A$ is *not* a non-crisp 2-element set.
- The condition on A means that it is:
  - not a 1-element set and
  - not a 2-element set.
- This means that there must exist at least three different values $x \in U$ for which $A(x) > 0$.
- For one of these values, we have $A(x_0) = 1$.
- Let us denote the other two values by x_1 and x_2, then $A(x_1) > 0$ and $A(x_2) > 0$.

40. Third Auxiliary Result (cont-d)

- Let us now take the following normalized fuzzy set B:
    - $B(x_1) = 0.5 \cdot A(x_1)$,
    - $B(x_2) = 0.5 \cdot A(x_2)$, and
    - $B(x) = A(x)$ for all other x.
- Here, $B(x_0) = A(x_0) = 1$, so B is indeed a normalized fuzzy set.
- One can easily check that $B(x) \leq A(x)$ for all x, so it is indeed a subset of A.
- Since $A(x_1) > 0$, we have $B(x_1) = 0.5 \cdot A(x_1) \neq A(x_1)$, so B is a proper subset of A.
- However, $B(x_0) = 1 > 0$, $B(x_1) > 0$, and $B(x_2) > 0$, so B is *not* a 2-element set.
- The proposition is proven.

41. Main Result

- **Proposition.** *A normalized fuzzy set is crisp if and only if we have one of the following two cases:*
  - *A is a 1-element fuzzy set, or*
  - *for every subset $B \subseteq A$ which is a non-crisp 2-element set, $\exists$ a crisp 2-element set C for which*

$$B \subseteq C \subseteq A.$$

- Let us first prove that if A is a crisp set, then:
  - either it is a 1-element crisp set,
  - or for every non-crisp 2-element set $B \subseteq A$, there exists a crisp 2-element set C for which $B \subseteq C \subseteq A$.
- Indeed, let B be a non-crisp 2-element set.

42. Main Result (cont-d)

- This means that for some elements $x_0 \in U$ and $x_1 \in U$, we have:
  - $B(x_0) = 1$,
  - $0 < B(x_1) < 1$, and
  - $B(x) = 0$ for all other x.
- Since $B \subseteq A$, we have:
  - $1 = B(x_0) \leq A(x_0)$ – thus $A(x_0) = 1$; and
  - $0 < B(x_2) \leq A(x_1)$ – thus $A(x_1) > 0$.
- The set A is crisp, so $A(x_1)$ can be either 0 or 1.
- Since $A(x_1) > 0$, we must have $A(x_1) = 1$.
- Thus, for a 2-element crisp set $C = \{x_0, x_1\}$, we have

$$B \subseteq C \subseteq A.$$

43. Main Result (cont-d)

- To complete our proof, let us prove that:
  - if a normalized crisp set A is *not* a crisp set,
  - then $\exists$ a non-crisp 2-element set $B \subseteq A$
  - for which no crisp 2-element set C satisfies the property $B \subseteq C \subseteq A$.
- By definition, for a crisp set, all the values $A(x)$ are either 0s or 1s.
- So, the fact that A is not crisp means that we have $0 < A(x_1) < 1$ for some $x_1 \in U$.
- Since A is normalized, $\exists x_0\,(A(x_0) = 1)$.

44. Main Result (cont-d)

- Let us now take the following set B:
  - $B(x_0) = 1$,
  - $0 < B(x_1) = A(x_1) < 1$, and
  - $B(x) = 0$ for all other x.
- Clearly, B is a non-crisp 2-element set and $B \subseteq A$.
- If we had $B \subseteq C \subseteq A$ for some crisp 2-element set C, then
  - due to $1 = B(x_0) \leq C(x_0)$ and $B(x_1) \leq C(x_1)$,
  - we would have $C(x_0) = 1$ and $C(x_1) > 0$ – hence $C(x_1) = 1$ (since C is crisp).
- But in this case, $C(x_1) = 1 > A(x_1)$, so we cannot have $C \subseteq A$.
- The proposition is proven.

45. Interval-Valued: First Auxiliary Result

- **Definition.** *By an* uncertain 1-element set, *we mean a normalized interval-valued fuzzy set A for which*

  $$\exists x_0 \in U\,(A(x_0) = [0,1]\,\&\,(A(x) = [0,0] \text{ for all other } x)).$$

- **Proposition.** *A normalized interval-valued fuzzy set A:*
  - *is an uncertain 1-element set if and only if*
  - *it has no proper normalized subsets.*
- Let us first prove that for an uncertain 1-element set A, there are no proper subsets.

46. Interval-Valued: 1st Auxiliary Result (cont-d)

- Indeed, if $A(x_0) = [0, 1]$, $A(x) = [0, 0]$ for all $x \neq x_0$, and $B(x) \leq A(x)$, then:
  - for $x \neq x_0$, from $\underline{B}(x) \leq \underline{A}(x) = 0$ and $\overline{B}(x) \leq \overline{A}(x) = 0$, it follows that $\underline{B}(x) = \overline{B}(x) = 0$, so
  $$B(x) = [0, 0] = A(x);$$
  - for $x = x_0$, from $\underline{A}(x_0) \leq \overline{A}(x_0) = 0$, it follows that
  $$\underline{B}(x_0) = 0 = \underline{A}(x_0).$$
- On the other hand, B is a normalized interval-valued fuzzy set, so we must have $\overline{B}(x) = 1$ for some x.
- This cannot be for $x \neq x_0$, since then $\overline{B}(x) = 0$.
- So, the only remaining option is $x = x_0$.
- Hence, $\overline{B}(x_0) = 1$, thus, $\overline{B}(x_0) = \overline{A}(x_0)$.

47. Interval-Valued: 1st Auxiliary Result (cont-d)

- Therefore, if $B \subseteq A$, then $B = A$.
- So, the normalized interval-valued fuzzy sets A does not have any proper subsets.
- To complete the proof, let us prove that:
  - if a normalized interval-valued fuzzy set has no proper subsets,
  - then it is an uncertain 1-element set.
- Indeed, since A is normalized, there exists an element x_0 for which $\overline{A}(x_0) = 1$.
- Then, as one can easily check, we have $B \subseteq A$, where:
  - $B(x_0) = [0, 1]$, and
  - $B(x) = [0, 0]$ for all other x
- Since A has no proper subsets, we thus conclude that $A = B$, i.e., that A is an uncertain 1-element set. QED

48. Interval-Valued: 2nd Auxiliary Result

- **Definition.** *By a* basic 1-element set, *we mean a normalized interval-valued fuzzy set A for which:*

$$\exists x_0 \in U\,((A(x_0) = [a, 1] \text{ for some } a > 0)\ \&$$

$$(A(x) = [0, 0] \text{ for all } x \neq x_0)).$$

- **Definition.** *By a* basic 2-element set, *we mean a norm. interval-valued fuzzy set A s.t. for some $x_0 \neq x_1$:*
  - $A(x_0) = [0, 1]$,
  - $A(x_1) = [0, a]$ *for some* $a \in (0, 1)$, *and*
  - $A(x) = [0, 0]$ *for all other* x.

49. Interval-Valued: 2nd Aux. Result (cont-d)

- **Proposition.**
  - *Let A be a normalized interval-valued fuzzy set which is not an uncertain 1-element set.*
  - *Then, the following two conditions are equivalent to each other:*
    - *the class $\{B : B \subseteq A\}$ of all subsets of A is linearly ordered;*
    - *A is either a basic 1-element set or a basic 2-element set.*
- Let us first prove that:
  - if A is a basic 1-element set or a basic 2-element set,
  - then the class of all its subsets is linearly ordered.

50. Interval-Valued: 2nd Aux. Result (cont-d)

- Let us first consider the case when A is a basic 1-element set.
- In this case, $B \subseteq A$ implies $\underline{B}(x) = \overline{B}(x) = 0$ for all $x \neq x_0$.
- Since B is normalized, then, similarly to the previous proofs, we get $\overline{B}(x_0) = 1$.
- The final inequality $\underline{B}(x_0) \leq \underline{A}(x_0) = a$ implies that for $b \stackrel{\text{def}}{=} \underline{B}(x_0)$, we have $b \leq a$.
- So, the set B has the following form:
  - $B(x) = [0, 0]$ for all $x \neq x_0$, and
  - $B(x_0) = [b, 1]$, where we denoted $b = \underline{B}(x_0)$.
- One can easily check that the class of such sets is linearly ordered.

51. Interval-Valued: 2nd Aux. Result (cont-d)

- Namely, if for two such sets B_1 and B_2, we denote the corresponding values b by b_1 and b_2, then:
  - if $b_1 \le b_2$, then $B_1 \subseteq B_2$, and
  - vice versa, if $b_2 \le b_1$, then $B_2 \subseteq B_1$.
- Let us consider the case when A is a basic 2-element set.
- Let $B \subseteq A$. Then, from $B(x) \le A(x)$, we conclude:
  - that $B(x) = [0, 0]$ when $x \neq x_0$ and $x \neq x_1$, and
  - that $\underline{B}(x_0) = \underline{B}(x_1) = 0$.
- The set B is normalized, so $\overline{B}(x) = 1$ for some x.
  - This x cannot be different from x_0 and x_1, since for such x, we have $\overline{B}(x) = 0 < 1$.
  - It cannot be equal to x_1, since we have

$$\overline{B}(x_1) \le \overline{A}(x_1) = a < 1.$$

52. Interval-Valued: 2nd Aux. Result (cont-d)

- Thus, the only possible element x is $x = x_0$, hence we have $\overline{B}(x_0) = 1$.
- The final inequality $\overline{B}(x_1) \le \overline{A}(x_1) = a$ implies that for $b \stackrel{\text{def}}{=} \overline{B}(x_1)$, we have $b \le a$.
- So, the set B has the following form:
  - $B(x) = [0, 0]$ when $x \ne x_0$ and $x \ne x_1$;
  - $B(x_0) = [0, 1]$, and
  - $B(x_1) = [0, b]$, where $b = \overline{B}(x_1)$.
- One can easily check that the class of such sets is linearly ordered.

53. Interval-Valued: 2nd Aux. Result (cont-d)

- Namely, if for two such sets B_1 and B_2, we denote the corresponding values b by b_1 and b_2, then:
  - if $b_1 \le b_2$, then $B_1 \subseteq B_2$, and
  - vice versa, if $b_2 \le b_1$, then $B_2 \subseteq B_1$.
- Let us now prove that:
  - if the class of all normalized subsets of a normalized fuzzy interval-valued set A is linearly ordered,
  - then A is either a basic 1-element set or a basic 2-element set.
- Since the set A is normalized, there exists an element $x_0 \in U$ for which $\overline{A}(x_0) = 1$.
- Let us consider two possible cases: $\underline{A}(x_0) > 0$ and $\underline{A}(x_0) = 0$.

54. Interval-Valued: 2nd Aux. Result (cont-d)

- Let us first consider he case when $\underline{A}(x_0) > 0$.
- Let us prove that in this case, we have a basic 1-element set, i.e., that $A(x) = [0, 0]$ for all $x \neq x_0$.
- We will prove this by contradiction.
- Let us assume that $\overline{A}(x) > 0$ for some $x \neq x_0$.
- Then, we can consider the following two subsets of A:
  - $B_1(x_0) = A(x_0)$, $B_2(x_0) = [0, 1]$;
  - $B_2(x_1) = [0, 0]$, $B_2(x_1) = A(x_1)$, and
  - $A(x) = B_i(x) = [0, 0]$ for al other $x \in U$.
- One can easily check that $B_1 \subseteq A$ and $B_2 \subseteq A$.

55. Interval-Valued: 2nd Aux. Result (cont-d)

- However:
  - we have $\underline{B}_1(x_0) = \underline{A}(x_0) > 0 = \underline{B}_2(x_0)$, hence we cannot have $B_1 \subseteq B_2$;
  - on the other hand, $\overline{B}_2(x_1) = \overline{A}(x_1) > 0 = \overline{B}_1(x_1)$, hence we cannot have $B_2 \subseteq B_1$.
- The fact that here $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$ shows that $\overline{A}(x) > 0$ is impossible.
- Thus, $\overline{A}(x) = 0$ for all $x \neq x_0$, so A is indeed a basic 1-element set.
- Let us now consider he case when $\underline{A}(x_0) = 0$.
- Let us prove that in this case, we have a basic 2-element set, i.e., that:
  - $A(x_1) = [0, a]$ for some $x_1 \in U$ and some $a \in (0, 1)$,
  - and $A(x) = [0, 0]$ for all other x.

56. Interval-Valued: 2nd Aux. Result (cont-d)

- Indeed, since $A(x_0) = [0,1]$, but the set A is not an uncertain 1-element set, $\exists x_1 \neq x_0\,(\overline{A}(x_1) > 0)$.
- Let us prove that in this case, $A(x) = [0,0]$ for all other x.
- We prove this by contradiction.
- Let us assume that for some x_2, we have $x_2 \neq x_0$, $x_2 \neq x_1$ and $\overline{A}(x_2) > 0$.
- In this case, we can form the following B_1 and B_2;
  - $B_1(x_0) = B_2(x_0) = [0,1]$;
  - $B_1(x_1) = A(x_1)$, $B_2(x_1) = [0,0]$;
  - $B_1(x_2) = [0,0]$, $B_2(x_2) = A(x_2)$; and
  - $B_1(x) = B_2(x) = [0,0]$ or all other x.

57. Interval-Valued: 2nd Aux. Result (cont-d)

- Clearly, $B_1 \subseteq A$ and $B_2 \subseteq A$, but:
  - $\overline{B}_1(x_1) > 0 = \overline{B}_2(x_1)$, so we cannot have $B_1 \subseteq B_2$;
  - $\overline{B}_2(x_2) = \overline{A}(x_2) > 0 = \underline{B}_1(x_2)$, so we cannot have $B_2 \subseteq B_1$.
- This contradicts to our assumption that the class of all subsets of A is linearly ordered.
- Thus, $A(x) = [0, 0]$ for all element x which are different from x_0 and x_1.
- Let us prove, by contradiction, that $\underline{A}(x_1) = 0$.
- Indeed, if $\underline{A}(x_1) > 0$, then we can form the following sets B_1 and B_2:
  - $B_1(x_0) = B_2(x_0) = [0, 1]$;
  - $B_1(x_1) = \left[0, \overline{A}(x_1)\right]$, $B_2(x_1) = 0.5 \cdot \underline{A}(x_1)$;
  - $B_1(x) = B_2(x) = [0, 0]$ for all other x.

58. Interval-Valued: 2nd Aux. Result (cont-d)

- One can easily check that $B_1 \subseteq A$ and $B_2 \subseteq A$, but:
  - $\overline{B}_1(x_1) = \overline{A}(x_1) \ge \underline{A}(x_1) > 0.5 \cdot \underline{A}(x_1) = \overline{B}_2(x_1)$, so we do not have $B_1 \subseteq B_2$;
  - on the other hand, $\underline{B}_2(x_1) = 0.5 \cdot \underline{A}(x_1) > 0 = \underline{B}_1(x_1)$, so we do not have $B_2 \subseteq B_1$ either.
- This contradicts to our assumption that the class of all subsets of A is linearly ordered.
- This contradiction shows that $\underline{A}_1(x_1) = 0$.
- Finally, let us prove that $\overline{A}(x_1) < 1$.

59. Interval-Valued: 2nd Aux. Result (cont-d)

- Indeed, if $\overline{A}(x_1) = 1$, i.e., if $A(x_1) = [0, 1]$, then we can find $B_1, B_2 \subseteq A$ for which $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$:
  - $B_1(x_0) = [0, 1]$, $B_2(x_0) = [0, 0]$;
  - $B_1(x_1) = [0, 0]$, $B_2(x_1) = A(x_1) = [0, 1]$, and
  - $B_1(x) = B_2(x) = [0, 0]$ for all other x.
- Then:
  - $\overline{B}_1(x_0) = 1 > \overline{B}_2(x_0)$, so we cannot have $B_1 \subseteq B_2$;
  - $\overline{B}_2(x_1) = 1 > 0 = \overline{B}_1(x_1)$, so $B_2 \not\subseteq B_1$.
- Contradiction show that we cannot have $\overline{A}(x_1) = 1$, thus $\overline{A}(x_1) < 1$.
- Thus, in this case, A is a basic 2-element set.
- The proposition is proven.

60. Interval-Valued: 3rd Auxiliary Result

- **Proposition.** *If A is a basic 1- or 2-element set, then the following properties are equivalent:*
  - *A is a crisp 1-element set;*
  - *no proper superset of A is a basic 1-element set or a basic 2-element set.*
- If $A = \{x_0\}$, then clearly A cannot have any proper supersets which are basic 1- or 2-element sets.
- Vice versa:
  - if A is a basic 1-element set with $\underline{A}(x_0) < 1$,
  - then $B = \{x_0\}$ is its proper superset which is a a 1-element basic set.

61. Interval-Valued: 3rd Aux. Result (cont-d)

- Similarly,
  - if A is a basic 2-element set, with $A(x_0) = [0, 1]$, $\underline{A}(x_1) = 0$, and $\overline{A}(x_1) < 1$,
  - then we can have the following proper superset $B \supseteq A$ which is also a basic 2-element set:
    - $B(x_0) = [0, 1]$;
    - $B(x_1) = \left[0, \dfrac{1 + \overline{A}(x_1)}{2}\right]$; and
    - $B(x) = 0$ for all other x.
- The proposition is proven.

62. Interval-Valued: 5th Auxiliary Result

- **Definition.**
  - *Let A be a basic 2-element set, with:*
    - $A(x_0) = [0, 1]$,
    - $A(x_1) = [0, a]$ *for some* $a \in (0, 1)$, *and*
    - $A(x) = [0, 0]$ *for all other* x.
  - *Then, by its* type-1 cover, *we mean a normalized interval-valued fuzzy set* A' *for which:*
    - $A'(x_0) = [1, 1]$,
    - $A'(x_1) = [a, a]$, *and*
    - $A'(x) = [0, 0]$ *for all other* x.
- Let us show that the type-1 cover can be determined in terms of the subsethood relation.

63. Interval-Valued: 5th Aux. Result (cont-d)

- **Proposition.** *Let A be a basic 2-element set. Then:*
  - *its type-1 cover A' is the $\subseteq$-smallest normalized interval-valued fuzzy set*
  - *that contains all the normalized interval-valued sets $B \supseteq A$ for which the following conditions hold:*
    - *the set B is not a basic 2-element set;*
    - *the class of all basic 2-element subsets of B is linearly ordered;*
    - *the class $\{C : C \text{ is normalized } \& \, A \subseteq C \subseteq B\}$ is linearly ordered; and*
    - *the set B has only one uncertain 1-element subset.*

64. Interval-Valued: 5th Aux. Result (cont-d)

- Let us first prove that B satisfies the above four conditions $\Leftrightarrow$ it has one the following 2 forms:
  - either it has the form $B(x_0) = [b, 1]$ for some $b > 0$, $B(x_1) = A(x_1)$, and $B(x) = [0, 0]$ for all other x;
  - we will call these B *of the first form*;
  - or it has the form $B(x_0) = A(x_0)$, $B(x_1) = [b, a]$ for some $b > 0$, and $B(x) = [0, 0]$ for all other x;
  - we will call these B *of the second form*.
- Let us first prove that the all the sets B of the first form satisfy all the above four conditions.
- Indeed, clearly, such B is not a basic 2-element set.

65. Interval-Valued: 5th Aux. Result (cont-d)

- If C is a basic 2-element set for which $C \subseteq B$, then we have:
  - $C(x_0) = [0, 1]$,
  - $C(x) = [0, 0]$ for all x different from x_0 and x_1, and
  - $C(x_1) = [0, c]$ for some $c \le a$.
- Clearly, the set of all such C is linearly ordered.
- Indeed, if we have two such sets, corresponding to elements c_1 and c_2, then:
  - if $c_1 \le c_2$, then we have $C_1 \subseteq C_2$, and
  - if $c_2 \le c_1$, then we have $C_2 \subseteq C_1$.

66. Interval-Valued: 5th Aux. Result (cont-d)

- If $A \subseteq C \subseteq B$, then we have:
  - $C(x_0) = [c, 1]$ for some $c \in [b, 1]$,
  - $C(x_1) = A(x_1)$, and
  - $C(x) = [0, 0]$ for all other x.
- Thus, if we have two such sets, corresponding to elements c_1 and c_2, then:
  - if $c_1 \leq c_2$, then we have $C_1 \subseteq C_2$, and
  - if $c_2 \leq c_1$, then we have $C_2 \subseteq C_1$.
- Of course, the only uncertain 1-element set contained in B is the set corresponding to x_0.
- All four conditions are proven.
- Let us now prove that the all the sets B of the second form satisfy all the above four conditions.

67. Interval-Valued: 5th Aux. Result (cont-d)

- Indeed, clearly, such B is not a basic 2-element set.
- If $C \subseteq B$ is a basic 2-element set, then we have:
  - $C(x_0) = [0, 1]$,
  - $C(x) = [0, 0]$ for all x different from x_0 and x_1, and
  - $C(x_1) = [0, c]$ for some $c \le a$.
- Clearly, the set of all such C is linearly ordered.
- Indeed, if we have two such sets, corresponding to elements c_1 and c_2, then:
  - if $c_1 \le c_2$, then we have $C_1 \subseteq C_2$, and
  - if $c_2 \le c_1$, then we have $C_2 \subseteq C_1$.

68. Interval-Valued: 5th Aux. Result (cont-d)

- If $A \subseteq C \subseteq B$, then we have:
  - $C(x_0) = A(x_0)$,
  - $C(x_1) = [c, a]$ for some $c \in [b, a]$, and
  - $C(x) = [0, 0]$ for all other x.
- Thus, if we have two such sets, corresponding to elements c_1 and c_2, then:
  - if $c_1 \leq c_2$, then we have $C_1 \subseteq C_2$, and
  - if $c_2 \leq c_1$, then we have $C_2 \subseteq C_1$.
- Of course, the only uncertain 1-element set contained in B is the set corresponding to x_0.
- All four conditions are proven.
- Let us now prove that if B satisfies the above conditions, then B is of the first or of the second form.

69. Interval-Valued: 5th Aux. Result (cont-d)

- Let us first prove that we must have $B(x) = [0, 0]$ for all elements x which are different from x_0 and x_1.
- We will prove this by contradiction.
- Assume that $\overline{B}(x_2) > 0$ for some element x_2 which is different from x_0 and x_1.
- Then, in addition to a basic 2-element set $A \subseteq B$, we also have another basic 2-element set $C \subseteq B$ for which:
  - $C(x_0) = [0, 1]$,
  - $C(x_2) = \left[0, \overline{B}(x_2)\right]$, and
  - $C(c) = [0, 0]$ for all other elements x.
- Then:
  - $\overline{A}(x_1) = a > 0 = \overline{C}(x_1)$, so $A \not\subseteq C$; and
  - $\overline{C}(x_2) > 0 = \overline{A}(x_2)$, so we cannot have $C \not\subseteq A$.

70. Interval-Valued: 5th Aux. Result (cont-d)

- This contradicts to the condition that set of all basic 2-element sets which are subsets of B is linearly ordered.
- Thus, $\overline{B}(x) > 0$ is impossible.
- So, indeed, $B(x) = [0, 0]$ for all elements x which are different from x_0 and x_1.
- Thus, the set B is uniquely described by its values $B(x_0)$ and $B(x_1)$.
- The condition that $A \subseteq B$ implies that $\overline{A}(x_0) = 1$ and that:
  - $\overline{B}(x_0) \geq 0$,
  - $\underline{B}(x_1) \geq 0$, and
  - that $\overline{B}(x_1) \geq a = \overline{A}(x_1)$.
- Since B is not a basic 2-element set and A is such a set, we have $B \neq A$.

71. Interval-Valued: 5th Aux. Result (cont-d)

- Thus, at least one of the above inequalities must be strict.
- Let us consider these three inequalities one by one.
- Let us first consider the case when $\overline{B}(x_0) > 0$.
- Let us prove that in this case, we have $B(x_1) = A(x_1)$, i.e., that we have a set of the first form.
- We will first prove, by contradiction, that $\underline{B}(x_1) = 0$.
- Indeed, if $\underline{B}(x_1) > 0$, then we can form C_1, C_2 for which $A \subseteq C_1 \subseteq B$, $A \subseteq C_2 \subseteq B$, $C_1 \not\subseteq C_2$, and $C_2 \not\subseteq C_1$:
  - $C_1(x_0) = A(x_0) = [0, 1]$, $C_1(x_1) = B(x_1)$, and $C_1(x) = [0, 0]$ for all other x;
  - $C_2(x_0) = B(x_0)$, $C_2(x_1) = A(x_1)$, and $C_2(x) = [0, 0]$ for all other x.

72. Interval-Valued: 5th Aux. Result (cont-d)

- Here:
  - $\underline{C}_1(x_1) = \underline{B}(x_1) > 0 = \underline{C}_2(x_1)$, so $C_1 \not\subseteq C_2$;
  - $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$, so $C_2 \not\subseteq C_1$.
- This contradicts to our assumption that the class of all intermediate fuzzy sets C is linearly ordered.
- Thus, we must have $\underline{B}(x_1) = 0$.
- Let us now prove, by contradiction, that $\overline{B}(x_1) = \overline{A}(x_1)$.
- Indeed, suppose that $\overline{B}(x_1) > \overline{A}(x_1)$.
- Then we can form C_1, C_2 for which $A \subseteq C_1 \subseteq B$, $A \subseteq C_2 \subseteq B$, $C_1 \not\subseteq C_2$, and $C_2 \not\subseteq C_1$:
  - $C_1(x_0) = A(x_0) = [0, 1]$, $C_1(x_1) = B(x_1)$, and $C_1(x) = [0, 0]$ for all other x;
  - $C_2(x_0) = B(x_0)$, $C_2(x_1) = A(x_1)$, and $C_2(x) = [0, 0]$ for all other x.

73. Interval-Valued: 5th Aux. Result (cont-d)

- Here:
  - $\overline{C}_1(x_1) = \overline{B}(x_1) > \overline{A}(x_1) = \overline{C}_2(x_1)$, so $C_1 \not\subseteq C_2$;
  - $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$, so $C_2 \not\subseteq C_1$.
- This contradicts to our assumption that the class of all intermediate fuzzy sets C is linearly ordered.
- Thus, we must have $\overline{B}(x_1) = \overline{A}(x_1)$.
- So, in this case, we indeed have a set of the first form.
- Let us now consider the case when $\underline{B}(x_1) > 0$.
- Let us prove that in this case, we have $\overline{B}(x_0) = 0$ and $\overline{B}(x_1) = \overline{A}(x_1)$.
- This would mean that we have a set of the second form.
- We will first prove, by contradiction, that $\underline{B}(x_0) = 0$.

74. Interval-Valued: 5th Aux. Result (cont-d)

- Indeed, if $\underline{B}(x_0) > 0$, then we can form C_1, C_2 for which $A \subseteq C_1 \subseteq B$, $A \subseteq C_2 \subseteq B$, $C_1 \not\subseteq C_2$, and $C_2 \not\subseteq C_1$:
  - $C_1(x_0) = A(x_0) = [0,1]$, $C_1(x_1) = B(x_1)$, and $C_1(x) = [0,0]$ for all other x;
  - $C_2(x_0) = B(x_0)$, $C_2(x_1) = A(x_1)$, and $C_2(x) = [0,0]$ for all other x.
- Here:
  - $\underline{C}_1(x_1) = \underline{B}(x_1) > 0 = \underline{C}_2(x_1)$, so $C_1 \not\subseteq C_2$;
  - $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$, so $C_2 \not\subseteq C_1$.
- This contradicts to our assumption that the class of all intermediate fuzzy sets C is linearly ordered.
- Thus, we must have $\underline{B}(x_0) = 0$.
- Let us now prove, by contradiction, that $\overline{B}(x_1) = \overline{A}(x_1)$.

75. Interval-Valued: 5th Aux. Result (cont-d)

- Indeed, suppose that $\overline{B}(x_1) > \overline{A}(x_1)$.
- Then we can form C_1, C_2 for which $A \subseteq C_1 \subseteq B$, $A \subseteq C_2 \subseteq B$, $C_1 \not\subseteq C_2$, and $C_2 \not\subseteq C_1$:
  - $C_1(x_0) = [0,1]$, $C_1(x_1) = B(x_1)$, and $C_1(x) = [0,0]$ for all other x;
  - $C_2(x_0) = B(x_0)$, $C_2(x_1) = A(x_1)$, and $C_2(x) = [0,0]$ for all other x.
- Here:
  - $\overline{C}_1(x_1) = \overline{B}(x_1) > \overline{A}(x_1) = \overline{C}_2(x_1)$, so $C_1 \not\subseteq C_2$;
  - $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$, so $C_2 \not\subseteq C_1$.
- This contradicts to our assumption that the class of all intermediate fuzzy sets C is linearly ordered.
- Thus, we must have $\overline{B}(x_1) = \overline{A}(x_1)$.

76. Interval-Valued: 5th Aux. Result (cont-d)

- So, in this case, we indeed have a set of the second form.
- Finally, let us prove that the case when $\overline{B}(x_1) > \overline{A}(x_1)$ is not possible.
- We will first prove, by contradiction, that in this case, $\underline{B}(x_0) = 0$.
- Indeed, if $\underline{B}(x_0) > 0$, then we can form C_1, C_2 for which $A \subseteq C_1 \subseteq B$, $A \subseteq C_2 \subseteq B$, $C_1 \not\subseteq C_2$, and $C_2 \not\subseteq C_1$:
  - $C_1(x_0) = A(x_0) = [0, 1]$, $C_1(x_1) = B(x_1)$, and $C_1(x) = [0, 0]$ for all other x;
  - $C_2(x_0) = B(x_0)$, $C_2(x_1) = A(x_1)$, and $C_2(x) = [0, 0]$ for all other x.

77. Interval-Valued: 5th Aux. Result (cont-d)

- Here:
  - $\overline{C}_1(x_1) = \overline{B}(x_1) > \overline{A}(x_1) = \overline{C}_2(x_1)$, so $C_1 \not\subseteq C_2$;
  - $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$, so $C_2 \not\subseteq C_1$.
- This contradicts to our assumption that the class of all intermediate fuzzy sets C is linearly ordered.
- Thus, we must have $\underline{B}(x_0) = 0$.
- Let us now prove, by contradiction, that $\underline{B}(x_1) = 0$.
- Indeed, suppose that $\underline{B}(x_1) > 0$.
- Then we can form C_1, C_2 for which $A \subseteq C_1 \subseteq B$, $A \subseteq C_2 \subseteq B$, $C_1 \not\subseteq C_2$, and $C_2 \not\subseteq C_1$:
  - $C_1(x_0) = A(x_0) = [0, 1]$, $C_1(x_1) = \left[0, \overline{B}(x_1)\right]$, and $C_1(x) = [0, 0]$ for all other x;
  - $C_2(x_0) = A(x_0) = [0, 1]$, $C_2(x_1) = \left[\underline{B}(x_1), \overline{A}(x_1)\right]$, and $C_2(x) = [0, 0]$ for all other x.

78. Interval-Valued: 5th Aux. Result (cont-d)

- Here:
  - $\overline{C}_1(x_1) = \overline{B}(x_1) > \overline{A}(x_1) = \overline{C}_2(x_1)$, so $C_1 \not\subseteq C_2$;
  - $\underline{C}_2(x_1) = \underline{B}(x_1) > 0 = \underline{C}_1(x_1)$, so $C_2 \not\subseteq C_1$.
- This contradicts to our assumption that the class of all intermediate fuzzy sets C is linearly ordered.
- Thus, we must have $\underline{B}(x_1) = 0$.
- Finally, $\overline{B}(x_1) < 1$, since otherwise B would have two uncertain 1-element subsets:
  - a subset corresponding to x_0, and
  - a subset corresponding to x_1,
- We know that $\overline{B}(x_0) = 1$ and we have proved that $\underline{B}(x_0) = \underline{B}(x_1) = 0$ and $\overline{B}(x_1) < 1$.
- So, we conclude that the set B is a basic 2-element set, but we explicitly assumed that it is not.

79. Interval-Valued: 5th Aux. Result (cont-d)

- Thus, the third inequality cannot be strict, so B is indeed either of the first form, or of the second form.
- One can check that the smallest set containing all such sets is indeed the set A'.
- The proposition is proven.

80. Interval-Valued: Main Result

- **Definition.**
  - *Let A be an uncertain 1-element set, with $A(x_0) = [0,1]$, and $A(x) = [0,0]$ for all other x.*
  - *Then, by its* type-1 cover, *we mean a crisp set*

$$A' = \{x_0\}.$$

- **Proposition.** *A normalized interval-valued fuzzy set is a type-1 set $\Leftrightarrow$ the following conditions hold:*
  - *if $B \subseteq A$ for some uncertain 1-element set, then $B' \subset A$, and*
  - *if $B \subseteq A$ for some basic 2-element set, then*

$$B' \subseteq A.$$

- One can see that the type-1 cover of a set $A(x) = [\underline{A}(x), \overline{A}(x)]$ has the form $A'(x) = \left[\overline{A}(x), \overline{A}(x)\right]$.

81. Interval-Valued: Main Result (cont-d)

- For a type-1 set, $\underline{A}(x) = \overline{A}(x)$, thus $A' = A$, and clearly, $A \subseteq B$ implies $A' \subseteq B$.
- Vice versa, let us prove that if the above two conditions are satisfied, then A is a type-1 set.
- In other words, let's prove that $\underline{A}(x) = \overline{A}(x)$ for all x.
- To prove this, let us consider two possible cases:
  - elements x for which $\overline{A}(x) = 1$, and
  - elements x for which $\overline{A}(x) < 1$.
- Let us first consider an element x for which

$$\overline{A}(x) = 1.$$

- In this case, $B \subseteq A$ for the uncertain 1-element set B for which $B(x) = [0, 1]$ and $B(y) = [0, 0]$ for all $y \neq x$.
- Then, $B' = \{x\}$, i.e., $B'(x) = [1, 1]$.

82. Interval-Valued: Main Result (cont-d)

- Thus, from $B' \subseteq A$ it follows that $1 = \underline{B}'(x) \le \underline{A}(x)$, so $\underline{A}(x) = 1 = \overline{A}(x)$.
- So, for such elements x, we indeed have $\underline{A}(x) = \overline{A}(x)$.
- Finally, let's consider an element x for which $\overline{A}(x) < 1$.
- Since A is normalized, there exists an element x_0 for which $\overline{A}(x_0) = 1$.
- Now, we can form the following basic 2-element set B: $B(x_0) = [0, 1]$, $B(x) = \left[0, \overline{A}(x)\right]$, and $B(y) = [0, 0]$ for all other elements y.
- Clearly, $B \subseteq A$, hence $B' \subseteq A$.
- Here, $B'(x) = \left[\overline{B}(x), \overline{B}(x)\right] = \left[\overline{A}(x), \overline{A}(x)\right]$.
- So, $B' \subseteq A$ implies $\underline{B}'(x) = \overline{A}(x) \le \underline{A}(x)$, thus

$$\underline{A}(x) = \overline{A}(x). \text{ QED}$$