Partial Orders for Representing Uncertainty, Causality, and Decision Making: General Properties, Operations, and Algorithms

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1. Partial Orders are Important

- One of the main objectives of science and engineering is to select the most beneficial decisions. For that:
  - we must know people's preferences,
  - we must have the information about different events (possible consequences of different decisions), and
  - since information is never absolutely accurate, we must have information about uncertainty.
- All these types of information naturally lead to partial orders:
  - For preferences, $a \preccurlyeq b$ means that b is preferable to a. This relation is used in decision theory.
  - For events, $a \preccurlyeq b$ means that a can influence b. This causality relation is used in space-time physics.
  - For uncertain statements, $a \preccurlyeq b$ means that a is less certain than b (fuzzy logic etc.).

2. Overview

- In each of the three areas, there is a lot of research about studying the corresponding partial orders.
- This research has revealed that some ideas are common in all three applications of partial orders.
- In our research, we analyze:
  - general properties, operations, and algorithms
  - related to partial orders for representing uncertainty, causality, and decision making.
- In our analysis, we will be most interested in uncertainty – the computer-science aspect of partial orders.
- In our presentation:
  - we first give a general outline,
  - then present the main algorithmic result in detail.

3. Brief Outline

- Introduction: partial orders are important
- *Uncertainty* is ubiquitous in applications of partial orders
- Original order relation and the uncertainty-motivated *experimentally confirmable* relation
- From *potentially* confirmable relation to *actually* confirmable one: extending Allen's interval algebra
- *Properties* of ordered spaces: when is the resulting ordered space a lattice
- How to *combine* ordered sets
- How to tell when a product of two partially ordered spaces has a certain *property*

4. Uncertainty is Ubiquitous in Applications of Partial Orders

- Uncertainty is explicitly mentioned only in the computer-science example of partial orders.
- However, uncertainty is ubiquitous in describing our knowledge about all three types of partial orders.
- For example, we may want to check what is happening exactly 1 second after a certain reaction.
- However, in practice, we cannot measure time exactly.
- So, we can only observe an event which is close to b – e.g., that occurs 1 ± 0.001 sec after the reaction.
- In general, we can only guarantee that the observed event is within a certain neighborhood U_b of the event b.
- In decision making, we similarly know the user's preferences only with some accuracy.

5. First Result: Possible and Necessary Orders

- Due to uncertainty, there is usually a whole class C of different ordering relations consistent with data.
- It is desirable to check when it is *possible* that $a \preccurlyeq b$, i.e., when $a\,r\,b$ for *some* r from a given class C of orders.
- It is desirable to check when it is *necessary* that $a \preccurlyeq b$, i.e., when $a\,r\,b$ for *all* r from a given class of orders.
- A relation $a\,R\,b$ is called a *possible order* if for some class C of orders, $a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b)$.
- A relation $a\,R\,b$ is called a *necessary order* if for some class C of orders, $a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b)$.
- **Theorem.** *R is a possible order $\Leftrightarrow$ R is reflexive.*
- **Theorem.** *R is a necessary order $\Leftrightarrow$ R is an order.*

6. Uncertainty-Motivated Experimentally Confirmable Relation

- Because of the uncertainty:
  - the only possibility to experimentally confirm that a precedes b (e.g., that a can causally influence b)
  - is when for some neighborhood U_b of the event b, we have $a \preccurlyeq \widetilde{b}$ for all $\widetilde{b} \in U_b$.
- In topological terms, this "experimentally confirmable" relation $a \prec b$ means that:
  - the element b is contained in the future cone $C_a^+ = \{c : a \preccurlyeq c\}$ of the event a
  - together with some neighborhood.
- In other words, b belongs to the *interior* K_a^+ of the closed cone C_a^+.
- Such relation, in which future cones are open, are called *open*.

7. Uncertainty-Motivated Experimentally Confirmable Relation (cont-d)

- In usual space-time models:
  - once we know the open cone K_a^+,
  - we can reconstruct the original cone C_a^+ as the closure of K_a^+: $C_a^+ = \overline{K_a^+}$.
- A natural question is: vice versa,
  - can we uniquely reconstruct an open order
  - if we know the corresponding closed order?
- In Chapter 3, we prove that this reconstruction is possible.
- This result provides a partial solution to a known open problem.

8. Reconstructing Open Order from the Closed Order

- A set X with a partial order $\prec$ is called a *kinematic space* if is satisfies the following conditions:

$$\forall a\,\exists a_-, a_+\,(a_- \prec a \prec a_+);\;\; \forall a, b\,(a \prec b \to \exists c\,(a \prec c \prec b));$$

$$\forall a, b, c\,(a \prec b, c \to \exists d\,(a \prec d \prec b, c));$$

$$\forall a, b, c\,(b, c \prec a \to \exists d\,(b, c \prec d \prec a)).$$

- A kinematic space is called *separable* if there exists a countable set $\{x_n\}$ such that

$$\forall a, b(a \prec b \Rightarrow \exists i\,(a \prec x_i \prec b)).$$

- For every separable kinematic space, we define *convergence* $s_n \to a$ as follows:

$$\forall a_-, a_+\,(a_- \prec a \prec a_+ \Rightarrow \exists N\,\forall n\,(n \geq N \Rightarrow a_- \prec s_n \prec a_+))).$$

9. Reconstructing Open Order from the Closed Order (cont-d)

- For each set S, its *closure* $\overline{S}$ is defined as the set of all the points a for which $s_n \to a$ for some $\{s_n\} \subseteq S$.
- A kinematic space is called *normal* if

$$b \in \overline{\{c : a \prec c\}} \Leftrightarrow a \in \overline{\{c : c \prec b\}}.$$

- This relation is called *closed order* and denoted by

$$a \preccurlyeq b.$$

- We say that a separable kinematic space is *complete* if every $\preccurlyeq$-decreasing bounded sequence has a limit.
- **Theorem.** *If* $\preccurlyeq = \preccurlyeq'$ *for two complete separable normal kinematic orders* $\prec$ *and* $\prec'$, *then* $\prec = \prec'$.

10. From Potentially Experimentally Confirmable (EC) Relation to Actually EC One

- It is also important to check what can be confirmed when we only have observations with a given accuracy.
- For example:
  - instead of the knowing the exact time location of an an event a,
  - we only know an event $\underline{a}$ that preceded a and an event $\overline{a}$ that follows a.
- In this case, the only information that we have about the actual event a is that it belongs to the interval

$$[\underline{a}, \overline{a}] \stackrel{\text{def}}{=} \{a : \underline{a} \preccurlyeq a \preccurlyeq \overline{a}\}.$$

- It is desirable to describe possible relations between such intervals.

11. From Potentially Experimentally Confirmable (EC) Relation to Actually EC One (cont-d)

- It is desirable to describe possible relations between such intervals.
- Such a description has already been done for intervals on the real line.
- The resulting description is known as Allen's algebra.
- In these terms, what we want is to generalize Allen's algebra to intervals over an arbitrary poset.
- Such a generalization is given in Chapter 4.
- Instead of intervals, we can also consider more general sets.
- Some preliminary results are also given in this dissertation.

12. Extending Allen's Result: Possible Relations Between Intervals in Partially Ordered Sets

Theorem. *For a combination of relations* $(r_{--}, r_{-+}, r_{+-}, r_{++})$, *the following two conditions are equivalent to each other:*

- *there exists a partially ordered set and values* $\underline{x} < \overline{x}$ *and* $\underline{y} < \overline{y}$ *from this set for which:*
  - r_{--} *is the relation between* $\underline{x}$ *and* $\underline{y}$,
  - r_{-+} *is the relation between* $\underline{x}$ *and* $\overline{y}$,
  - r_{+-} *is the relation between* $\overline{x}$ *and* $\underline{y}$, *and*
  - r_{++} *is the relation between* $\overline{x}$ *and* $\overline{y}$.
- *the combination* $(r_{--}, r_{-+}, r_{+-}, r_{++})$ *is equal to one of the following combinations:*

  $$(<,<,<,<), (<,<,=,<), (<,<,\|,<), (<,<,>,<),$$

  $$(<,<,\|,\|),, (<,<,>,=), (<,<,>,\|), (<,<,>,>), \ldots$$

  (full list is given in Chapter 4).

13. Properties of Ordered Spaces

- Once a new ordered set is defined, we may be interested in its properties.
- For example, we may want to know when such an order is a lattice, i.e., when:
  - for every two elements,
  - there is the greatest lower bound and the least upper bound.
- If this set is not a lattice, we may want to know:
  - when the order is a *semi-lattice*, i.e., e.g.,
  - when every two elements have the least upper bound.
- The lattice property is analyzed in Chapter 5.
- In particular, we describe when special relativity-type ordered spaces are lattices.

14. When Special-Relativity-Type Spaces Are Lattices

- Let X be a metric space with distance d.
- A set $\mathbb{R}\times X$ with an ordering relation $(t,x)\preccurlyeq(s,y)\Leftrightarrow s-t\ge d(x,y)$ is called a *Busemann product.*
- A *geodesic arc* connecting two points x,y is a set of points x_α for which $x_0=x$, $x_1=y$, and

$$d(x_\alpha,x_\beta)=|\alpha-\beta|.$$

- A metric space is called a *real tree* if its every two points can be connected by exactly one geodesic arc.
- **Theorem.** *For each metric space X, the following conditions are equivalent to each other:*
  - *the Busemann product $\mathbb{R}\times X$ is a lattice;*
  - *the space X is a real tree.*

15. Towards Combining Ordered Spaces: Fuzzy Logic

- In the traditional 2-valued logic, every statement is either true or false.
- Thus, the set of possible truth values consists of two elements: true (1) and false (0).
- Fuzzy logic takes into account that people have different degrees of certainty in their statements.
- Traditionally, fuzzy logic uses values from the interval $[0, 1]$ to describe uncertainty.
- In this interval, the order is *total* (*linear*) in the sense that for every $a, a' \in [0, 1]$, either $a \preccurlyeq a'$ or $a' \preccurlyeq a$.
- However, often, *partial* orders provide a more adequate description of the expert's degree of confidence.

16. Towards General Partial Orders

- For example, an expert cannot describe her degree of certainty by an exact number.
- Thus, it makes sense to describe this degree by an *interval* $[\underline{d}, \overline{d}]$ of possible numbers.
- Intervals are only partially ordered; e.g., the intervals $[0.5, 0.5]$ and $[0, 1]$ are not easy to compare.
- More complex sets of possible degrees are also sometimes useful.
- Not to miss any new options, in this section, we consider general partially ordered spaces.

17. Need for Product Operations

- Often, two (or more) experts evaluate a statement S.
- Then, our certainty in S is described by a pair (a_1, a_2), where $a_i \in A_i$ is the i-th expert's degree of certainty.
- To compare such pairs, we must therefore define a partial order on the set $A_1 \times A_2$ of all such pairs.
- One example of a partial order on $A_1 \times A_2$ is a *Cartesian* product: $(a_1, a_2) \preccurlyeq (a'_1, a'_2) \Leftrightarrow ((a_1 \preccurlyeq a'_1) \,\&\, (a_2 \preccurlyeq a'_2))$.
- This is a *cautious* approach, when our confidence in S' is higher than in $S \Leftrightarrow$ it is higher for both experts.
- *Lexicographic* product: $(a_1, a_2) \preccurlyeq (a'_1, a'_2) \Leftrightarrow$

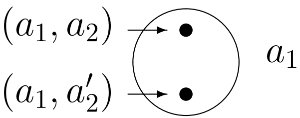
$$((a_1 \preccurlyeq a'_1) \,\&\, a_1 \neq a'_1) \vee ((a_1 = a'_1) \,\&\, (a_2 \preccurlyeq a'_2))).$$

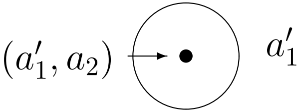
- Here, we are absolutely confident in the 1st expert – and only use the 2nd when the 1st is not sure.

18. Possible Physical Meaning of Lexicographic Order

Idea:

- A_1 is *macroscopic* space-time,
- A_2 is *microscopic* space-time:





19. Products of Ordered Sets: What Is Known

- At present, two product operations are known:
  - *Cartesian* product

$$(a_1, a_2) \preccurlyeq (a'_1, a'_2) \Leftrightarrow (a_1 \preccurlyeq_1 a'_1 \,\&\, a_2 \preccurlyeq_2 a'_2);$$

  and

  - *lexicographic* product

$$(a_1, a_2) \preccurlyeq (a'_1, a'_2) \Leftrightarrow$$

$$((a_1 \preccurlyeq_1 a'_1 \,\&\, a_1 \neq a'_1) \vee (a_1 = a'_1 \,\&\, a_2 \preccurlyeq_2 a'_2).$$

- *Question:* what other operations are possible?

20. Describing All Possible Products: A Theorem

- By a *product operation*, we mean a Boolean function

$$P : \{T, F\}^4 \to \{T, F\}.$$

- For every two partially ordered sets A_1 and A_2, we define the following relation on $A_1 \times A_2$:

$$(a_1, a_2) \preccurlyeq (a'_1, a'_2) \stackrel{\text{def}}{=}$$

$$P(a_1 \preccurlyeq_1 a'_1, a'_1 \preccurlyeq_1 a_1, a_2 \preccurlyeq_2 a'_2, a'_2 \preccurlyeq_2 a_2).$$

- We say that a product operation is *consistent* if $\preccurlyeq$ is always a partial order, and

$$(a_1 \preccurlyeq_1 a'_1 \,\&\, a_2 \preccurlyeq_2 a'_2) \Rightarrow (a_1, a_2) \preccurlyeq (a'_1, a'_2).$$

- **Theorem:** *Every consistent product operation is the Cartesian or the lexicographic product.*

21. Products: Natural Questions

- *Question:* when does the resulting partially ordered set $A_1 \times A_2$ satisfy a certain property?
- *Examples:* is it a total order? is it a lattice order?
- *It is desirable* to reduce the question about $A_1 \times A_2$ to questions about properties of component spaces A_i.
- *Some such reductions are known*; e.g.:
  - A Cartesian product is a total order $\Leftrightarrow$ one of A_i is a total order, and the other has only one element.
  - A lexicographic product is a total order if and only if both components are totally ordered.
- In this dissertation, we provide a general algorithm for such reduction.

22. Similar Questions in Other Areas

- Similar questions arise in *other applications* of ordered sets.
- *Example:* in space-time geometry, $a \preccurlyeq b$ means that an event a can influence the event b.
- *Our algorithm* does not use the fact that the original relations are orders.
- Thus, our algorithm is applicable to a *general binary relation* – equivalence, similarity, etc.
- Moreover, this algorithm can be applied to the case when we have a space with *several* binary relations.
- *Example:* we may have an order relation and a similarity relation.

23. Definitions

- *By a* space, *we mean a set A with m binary relations $P_1(a, a'), \ldots, P_m(a, a')$.*
- *By a* 1st order property, *we mean a formula F obtained from $P_i(x, x')$ by using logical $\vee$, $\&$, $\neg$, $\rightarrow$, $\exists x$ and $\forall x$.*
- *Note:* most properties of interest are 1st order; e.g. to be a total order means $\forall a \forall a' \, ((a \preccurlyeq a') \vee (a' \preccurlyeq a))$.
- *By a* product operation, *we mean a collection of m propositional formulas that*
  - *describe the relation $P_i((a_1, a_2), (a'_1, a'_2))$ between the elements $(a_1, a_2), (a'_1, a'_2) \in A_1 \times A_2$*
  - *in terms of the relations between the components $a_1, a'_1 \in A_1$ and $a_2, a'_2 \in A_2$ of these elements.*
- *Note:* both Cartesian and lexicographic order are product operations in this sense.

24. Combining Orders: Main Result

- **Theorem.** *There exists an algorithm that, given*
  - *a product operation and*
  - *a property F,*

  generates a list of properties F_{11}, F_{12}, ..., F_{p1}, F_{p2} s.t.:

$$F(A_1 \times A_2) \Leftrightarrow ((F_{11}(A_1) \,\&\, F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1) \,\&\, F_{p2}(A_2))).$$

- *Example:* For Cartesian product and total order F, we have

  $$F(A_1 \times A_2) \Leftrightarrow ((F_{11}(A_1) \,\&\, F_{12}(A_2)) \vee (F_{21}(A_1) \,\&\, F_{22}(A_2))):$$

  - $F_{11}(A_1)$ means that A_1 is a total order,
  - $F_{12}(A_2)$ means that A_2 is a one-element set,
  - $F_{21}(A_1)$ means that A_1 is a one-element set, and
  - $F_{22}(A_2)$ means that A_2 is a total order.

25. Proof of the Main Result

- The desired property $F(A_1 \times A_2)$ uses:
  - relations $P_i(a, a')$ between elements $a, a' \in A_1 \times A_2$;
  - quantifiers $\forall a$ and $\exists a$ over elements $a \in A_1 \times A_2$.
- Every element $a \in A_1 \times A_2$ is, by definition, a pair (a_1, a_2) in which $a_1 \in A_1$ and $a_2 \in A_2$.
- Let us explicitly replace each variable with such a pair.
- By definition of a product operation:
  - each relation $P_i((a_1, a_2), (a'_1, a'_2))$
  - is a propositional combination of relations betw. elements $a_1, a'_1 \in A_1$ and betw. elements $a_2, a'_2 \in A_2$.
- Let us perform the corresponding replacement.
- Each quantifier can be replaced by quantifiers corresponding to components: e.g., $\forall (a_1, a_2) \Leftrightarrow \forall a_1 \forall a_2$.

26. Proof of the Main Result (cont-d)

- So, we get an equivalent reformulation of F s.t.:
  - elementary formulas are relations between elements of A_1 or between A_2, and
  - quantifiers are over A_1 or over A_2.
- We use induction to reduce to the desired form

$$((F_{11}(A_1) \,\&\, F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1) \,\&\, F_{p2}(A_2))).$$

- Elementary formulas are already of the desired form – provided, of course, that we allow free variables.
- We will show that:
  - if we apply a propositional connective or a quantifier to a formula of this type,
  - then we can reduce the result again to the formula of this type.

27. Applying Propositional Connectives

- We apply propositional connectives to formulas of the type

$$((F_{11}(A_1) \,\&\, F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1) \,\&\, F_{p2}(A_2))).$$

- We thus get a propositional combination of the formulas of the type $F_{ij}(A_j)$.
- An arbitrary propositional combination can be described as a disjunction of conjunctions (DNF form).
- Each conjunction combines properties related to A_1 and properties related to A_2, i.e., has the form

$$G_1(A_1) \,\&\, \ldots \,\&\, G_p(A_1) \,\&\, G_{p+1}(A_2) \,\&\, \ldots \,\&\, G_q(A_2).$$

- Thus, each conjunction has the from $G(A_1) \,\&\, G'(A_2)$, where $G(A_1) \Leftrightarrow (G_1(A_1) \,\&\, \ldots \,\&\, G_p(A_1))$.
- Thus, the disjunction of such properties has the desired form.

28. Applying Existential Quantifiers

- When we apply $\exists a_1$, we get a formula
$$\exists a_1\,((F_{11}(A_1)\,\&\,F_{12}(A_2))\vee\ldots\vee(F_{p1}(A_1)\,\&\,F_{p2}(A_2))).$$
- It is known that $\exists a\,(A\vee B)$ is equivalent to $\exists a\,A\vee\exists a\,B$.
- Thus, the above formula is equivalent to a disjunction
$$\exists a_1\,(F_{11}(A_1)\,\&\,F_{12}(A_2))\vee\ldots\vee\exists a_1\,(F_{p1}(A_1)\,\&\,F_{p2}(A_2)).$$
- Thus, it is sufficient to prove that each formula $\exists a_1\,(F_{i1}(A_1)\,\&\,F_{i2}(A_2))$ has the desired form.
- The term $F_{i2}(A_2)$ does not depend on a_1 at all, it is all about elements of A_2.
- Thus, the above formula is equivalent to
$$(\exists a_1\,F_{i1}(A_1))\,\&\,F_{i2}(A_2).$$
- So, it is equivalent to the formula $F'_{i1}(A_1)\,\&\,F_{i2}(A_2)$, where $F'_{i1}\Leftrightarrow\exists a_1\,F_{i1}(A_1)$.

29. Applying Universal Quantifiers

- When we apply a universal quantifier, e.g., $\forall a_1$, then we can use the fact that $\forall a_1\, F$ is equivalent to $\neg\exists a_1\, \neg F$.
- We assumed that the formula F is of the desired type

$$(F_{11}(A_1)\,\&\,F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1)\,\&\,F_{p2}(A_2)).$$

- By using the propositional part of this proof, we conclude that $\neg F$ can be reduced to the desired type.
- Now, by applying the $\exists$ part of this proof, we conclude that $\exists a_1\, (\neg F)$ can also be reduced to the desired type.
- By using the propositional part again, we conclude that $\neg(\exists a_1\, \neg F)$ can be reduced to the desired type.
- By induction, we can now conclude that the original formula can be reduced to the desired type.
- The main result is proven.

30. Example of Applying the Algorithm

- Let us apply our algorithm to checking whether a Cartesian product is totally ordered.
- In this case, F has the form $\forall a \forall a' \,((a \preccurlyeq a') \vee (a' \preccurlyeq a))$.
- We first replace each variable $a, a' \in A_1 \times A_2$ with the corresponding pair:

$$\forall (a_1, a_2) \forall (a'_1, a'_2) \,(((a_1, a_2) \preccurlyeq (a'_1, a'_2)) \vee ((a'_1, a'_2) \preccurlyeq (a_1, a_2))).$$

- Replacing the ordering relation on the Cartesian product with its definition, we get

$$\forall (a_1, a_2) \forall (a'_1, a'_2) \,((a_1 \preccurlyeq a'_1 \,\&\, a_2 \preccurlyeq a'_2) \vee (a'_1 \preccurlyeq a_1 \,\&\, a'_2 \preccurlyeq a_2)).$$

- Replacing $\forall a$ over pairs with individual $\forall a_i$, we get:

$$\forall a_1 \forall a_2 \forall a'_1 \forall a'_2 \,((a_1 \preccurlyeq a'_1 \,\&\, a_2 \preccurlyeq a'_2)) \vee ((a'_1 \preccurlyeq a_1 \,\&\, a'_2 \preccurlyeq a_2))).$$

- By using the $\forall \Leftrightarrow \neg \exists \neg$, we get an equivalent form

$$\neg \exists a_1 \exists a_2 \exists a'_1 \exists a'_2 \,\neg((a_1 \preccurlyeq a'_1 \,\&\, a_2 \preccurlyeq a'_2) \vee (a'_1 \preccurlyeq a_1 \,\&\, a'_2 \preccurlyeq a_2))).$$

31. Example (cont-d)

- So far, we got:

$$\neg\exists a_1\exists a_2\exists a_1'\exists a_2'\,\neg((a_1 \preccurlyeq a_1' \,\&\, a_2 \preccurlyeq a_2')\vee(a_1' \preccurlyeq a_1 \,\&\, a_2' \preccurlyeq a_2))).$$

- Moving $\neg$ inside the propositional formula, we get

$$\neg\exists a_1\exists a_1\exists a_1'\exists a_2'\,((a_1 \not\preccurlyeq a_1' \vee a_2 \not\preccurlyeq a_2')\,\&\,(a_1' \not\preccurlyeq a_1 \vee a_2' \not\preccurlyeq a_2))).$$

- The formula $(a_1 \not\preccurlyeq a_1' \vee a_2 \not\preccurlyeq a_2')\,\&\,(a_1' \not\preccurlyeq a_1 \vee a_2' \not\preccurlyeq a_2)$ must now be transformed into a DNF form.
- The result is $(a_1 \not\preccurlyeq a_1' \,\&\, a_1' \not\preccurlyeq a_1)\vee(a_1 \not\preccurlyeq a_1' \,\&\, a_2' \not\preccurlyeq a_2)\vee(a_2 \not\preccurlyeq a_2' \,\&\, a_1' \not\preccurlyeq a_1) \vee (a_2 \not\preccurlyeq a_2' \,\&\, a_2' \not\preccurlyeq a_2)$.
- Thus, our formula is $\Leftrightarrow \neg(F_1 \vee F_2 \vee F_3 \vee F_4)$, where

$$F_1 \Leftrightarrow \exists a_1\exists a_2\exists a_1'\exists a_2'\,(a_1 \not\preccurlyeq a_1' \,\&\, a_1' \not\preccurlyeq a_1),$$

$$F_2 \Leftrightarrow \exists a_1\exists a_2\exists a_1'\exists a_2'\,(a_1 \not\preccurlyeq a_1' \,\&\, a_2' \not\preccurlyeq a_2),$$

$$F_3 \Leftrightarrow \exists a_1\exists a_2\exists a_1'\exists a_2'\,(a_2 \not\preccurlyeq a_2' \,\&\, a_1' \not\preccurlyeq a_1),$$

$$F_4 \Leftrightarrow \exists a_1\exists a_2\exists a_1'\exists a_2'\,(a_2 \not\preccurlyeq a_2' \,\&\, a_2' \not\preccurlyeq a_2).$$

32. Example (cont-d)

- So far, we got $\Leftrightarrow \neg(F_1 \vee F_2 \vee F_3 \vee F_4)$, where

$$F_1 \Leftrightarrow \exists a_1 \exists a_2 \exists a'_1 \exists a'_2\, (a_1 \not\precsim a'_1 \,\&\, a'_1 \not\precsim a_1),$$

$$F_2 \Leftrightarrow \exists a_1 \exists a_2 \exists a'_1 \exists a'_2\, (a_1 \not\precsim a'_1 \,\&\, a'_2 \not\precsim a_2),$$

$$F_3 \Leftrightarrow \exists a_1 \exists a_2 \exists a'_1 \exists a'_2\, (a_2 \not\precsim a'_2 \,\&\, a'_1 \not\precsim a_1),$$

$$F_4 \Leftrightarrow \exists a_1 \exists a_2 \exists a'_1 \exists a'_2\, (a_2 \not\precsim a'_2 \,\&\, a'_2 \not\precsim a_2).$$

- By applying the quantifiers to the corresponding parts of the formulas, we get

$$F_1 \Leftrightarrow \exists a_1 \exists a'_1\, (a_1 \not\precsim a'_1 \,\&\, a'_1 \not\precsim a_1),$$

$$F_2 \Leftrightarrow (\exists a_1 \exists a'_1\, a_1 \not\precsim a'_1) \,\&\, (\exists a_2 \exists a'_2\, a'_2 \not\precsim a_2),$$

$$F_3 \Leftrightarrow (\exists a_1 \exists a'_1\, a'_1 \not\precsim a_1) \,\&\, (\exists a_2 \exists a'_2\, a_2 \not\precsim a'_2),$$

$$F_4 \Leftrightarrow \exists a_2 \exists a'_1 \exists a'_2\, (a_2 \not\precsim a'_2 \,\&\, a'_2 \not\precsim a_2).$$

- Then, we again reduce $\neg(F_1 \vee F_2 \vee F_3 \vee F_4)$ to DNF.

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34. My Publications

- H.-P. A. Künzi, F. Zapata, and V. Kreinovich, "When is the Busemann product a lattice? A relation between metric spaces and corresponding space-time models", *Applied Mathematical Sciences*, 2012, Vol. 6, No. 66, pp. 3267–3276.
- F. Zapata, "Modal intervals as a new logical interpretation of the usual lattice order between interval truth values", *Proceedings of the Annual Conference of the North American Fuzzy Information Processing Society NAFIPS'2012*, Berkeley, California, August 6–8, 2012.
- F. Zapata and O. Kosheleva, "Possible and necessary orders, equivalences, etc.: From modal logic to modal mathematics", *Journal of Uncertain Systems*, 2013, Vol. 7, to appear.

35. My Publications (cont-d)

- F. Zapata, O. Kosheleva, and K. Villaverde, "Products of Partially Ordered Sets (Posets) and Intervals in Such Products, with Potential Applications to Uncertainty Logic and Space-Time Geometry", *Abstracts of the 14th GAMM-IMACS International Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics SCAN'2010*, Lyon, France, September 27–30, 2010, pp. 142–144.
- F. Zapata, O. Kosheleva, and K. Villaverde, "How to tell when a product of two partially ordered spaces has a certain property: General results with application to fuzzy logic", *Proceedings of the 30th Annual Conference of the North American Fuzzy Information Processing Society NAFIPS'2011*, El Paso, Texas, March 18–20, 2011.

36. My Publications (cont-d)

- F. Zapata, O. Kosheleva, and K. Villaverde, "How to tell when a product of two partially ordered spaces has a certain property?", *Journal of Uncertain Systems*, 2012, Vol. 6, No. 2, pp. 152–160.
- F. Zapata, O. Kosheleva, and K. Villaverde, "Product of partially ordered sets (posets), with potential applications to uncertainty logic and space-time geometry", *International Journal of Innovative Management, Information & Production (IJIMIP)*, to appear.
- F. Zapata and V. Kreinovich, "Reconstructing an open order from its closure, with applications to space-time physics and to logic", *Studia Logica*, 2012, Vol. 100, No. 1–2, pp. 419–435.

37. My Publications (cont-d)

- F. Zapata, V. Kreinovich, C. Joslyn, and E. Hogan, "Orders on Intervals Over Partially Ordered Sets: Extending Allen's Algebra and Interval Graph Results", *Soft Computing*, to appear.
- F. Zapata, E. Ramirez, J. A. Lopez, and O. Kosheleva, "Strings lead to lattice-type causality", *Journal of Uncertain Systems*, 2011, Vol. 5, No. 2, pp. 154–160.

38. Case Study: Lattice Order in Fuzzy Logic

- *Traditionally:* fuzzy logic uses numbers $d \in [0,1]$ as truth values.
- These numbers are *easy to compare*: if $d < d'$, this means more confidence in the statement S' than in S.
- One way to get the value d is by *polling:* if m out of n experts believe in S, take $d = m/n$.
- *Problem:*
  - if 4 out of 5 believe in S, we take $d = 4/5 = 0.8$, but
  - if we ask the 6th person, we never get 0.8 as $m/6$.
- *Solution:* instead of a single number d, use an interval $[\underline{d}, \overline{d}] \subseteq [0,1]$ of possible values of d.
- *Challenge:* how to compare different intervals?
- *Example:* how to compare $[0,1]$ and $[0.5, 0.5]$?

39. Extending $<$ from Numbers to Intervals as a Particular Case of a General Problem

- *How to* extend an order between numbers to intervals?
- Such problems are *typical* in fuzzy computations:
  - we have a f-n $f(x_1, \ldots, x_n)$ defined for real numbers,
  - we need to extend it to fuzzy numbers $X_1, \ldots, X_n$ (e.g., to intervals).
- *Solution:* Zadeh's extension principle (ZEP).
- For *intervals:* according to Zadeh's EP, we return the range of all possible values of $f(x_1, \ldots, x_n)$:

$$f(X_1, \ldots, X_n) \stackrel{\text{def}}{=} \{f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}.$$

- The task of computing this range for different $f(x_1, \ldots, x_n)$ and X_i constitutes *interval computations*.

40. Zadeh's Extension Principle Approach Applied to the Original Ordering Relation $\leq$

- There are three possible situations:
  - every $a \in [\underline{a}, \overline{a}]$ is smaller than or equal than every $b \in [\underline{b}, \overline{b}]$; then, the set $\leq(\mathbf{a}, \mathbf{b}) = \{1\}$ ("true");
  - if none of $a \in [\underline{a}, \overline{a}]$ is smaller than or equal than any $b \in [\underline{b}, \overline{b}]$, then $\leq(\mathbf{a}, \mathbf{b}) = \{0\}$ ("false");
  - in all other case, the set $\leq(\mathbf{a}, \mathbf{b})$ contains both 1 and 0, i.e., we have $\leq(\mathbf{a}, \mathbf{b}) = \{0, 1\}$.
- So, $\leq(\mathbf{a}, \mathbf{b})$ is true if and only if

$$\forall a \in \mathbf{a}\, \forall b \in \mathbf{b}\, (a \leq b).$$

- This is, in turn, equivalent to

$$\leq\left([\underline{a}, \overline{a}], \left[\underline{b}, \overline{b}\,\right]\right) \Leftrightarrow \overline{a} \leq \underline{b}.$$

41. Zadeh's Extension Principle Applied to the Function $\max(a, b)$

- $\max(a, b)$ is non-strictly increasing in a and b.
- Thus, when $a \in [\underline{a}, \overline{a}]$, and $b \in [\underline{b}, \overline{b}]$:
  - the smallest possible value of $\max(a, b)$ is attained when both a and b are the smallest:
  $$a = \underline{a} \text{ and } b = \underline{b};$$
  - the largest possible value of $\max(a, b)$ is attained when both a and b are the largest: $a = \overline{a}$ and $b = \overline{b}$;
- So, $\max([\underline{a}, \overline{a}], [\underline{b}, \overline{b}]) = [\max(\underline{a}, \underline{b}), \max(\overline{a}, \overline{b})]$.
- Now the relation $\mathbf{a} \le \mathbf{b}$, defined as $\mathbf{b} = \max(\mathbf{a}, \mathbf{b})$, takes the form $\underline{a} \le \underline{b}$ and $\overline{a} \le \overline{b}$.
- This relation is actively used in interval-valued fuzzy logic.

42. What We Do

- For the ordering relation $\overline{a} \le \underline{b}$ (obtained by applying Zadeh's EP to $\le$) we have a *logical* interpretation.
- The relation $\underline{a} \le \underline{b}$ and $\overline{a} \le \overline{b}$ coming from $\max(a, b)$ is *different*.
- Operations $\max(a, b)$ and $\min(a, b)$ form a lattice, so this relation is called a *lattice order*.
- *Problem:* how to interpret the lattice order in logical terms?
- *In this paper:* we provide the desired logical explanation for the lattice order.
- For that, we use *modal intervals*, a practice-motivated generalization of intervals.

43. Modal Intervals: A Brief Reminder

- *Traditional interval computations:*
  - we know the intervals $X_1, \ldots, X_n$ containing $x_1, \ldots, x_n$;
  - we know that a quantity z depends on $x = (x_1, \ldots, x_n)$:

$$z = f(x_1, \ldots, x_n);$$

  - we want to find the range Z of possible values of z:

$$Z = \left[\min_{x \in X} f(x), \max_{x \in X} f(x)\right].$$

- *Control situations:*
  - the value $z = f(x, u)$ also depends on the control variables $u = (u_1, \ldots, u_m)$;
  - we want to find Z for which, for every $x_i \in X_i$, we can get $z \in Z$ by selecting appropriate $u_j \in U_j$:

$$\forall x\, \exists u\, (z = f(x, u) \in Z).$$

44. Reformulation in Logical Terms – of Modal Intervals

- *Reminder:* we want $\forall x_{\in X}\, \exists u_{\in U}\, (f(x,u) \in Z)$.
- There is a logical difference between intervals X and U.
- The property $f(x,u) \in Z$ must hold
  - *for all* possible values $x_i \in X_i$, but
  - *for some* values $u_j \in U_j$.
- We can thus consider pairs of intervals and quantifiers (*modal intervals*):
  - each original interval X_i is a pair $\langle X_i, \forall\rangle$, while
  - controlled interval is a pair $\langle U_j, \exists\rangle$.
- We can treat the resulting interval Z as the range defined over modal intervals:

$$Z = f(\langle X_1, \forall\rangle, \ldots, \langle X_n, \forall\rangle, \langle U_1, \exists\rangle, \ldots, \langle U_m, \exists\rangle).$$

45. Modal Intervals Explain Lattice Order

- The relation $\overline{a} \le \underline{b}$ means that

$$\forall a \in \mathbf{a}\, \forall b \in \mathbf{b}\, (a \le b).$$

- This corresponds to *traditional* interval computation with $\forall$-intervals $\mathbf{a}$ and $\mathbf{b}$.
- If we replace one of the traditional $\forall$-intervals with the *modal* $\exists$-interval, we get two formulas:

$$\forall a \in \mathbf{a}\, \exists b \in \mathbf{b}\, (a \le b) \quad (1)$$

$$\forall b \in \mathbf{b}\, \exists a \in \mathbf{a}\, (a \le b). \quad (2)$$

- One can prove that:
  - the first formula is equivalent to $\overline{a} \le \overline{b}$; and
  - the second formula is equivalent to $\underline{a} \le \underline{b}$.
- Thus, modal intervals indeed explain lattice order.

46. Proof: First Formula

$$\forall a \in \mathbf{a}\, \exists b \in \mathbf{b}\, (a \le b)$$

- If $a \le b$ for some b for which $\underline{b} \le b \le \overline{b}$ then, by transitivity, we get $a \le \overline{b}$.
- Vice versa, if $a \le \overline{b}$, then $a \le b$ for some $b \in [\underline{b}, \overline{b}]$: namely, for $b = \overline{b}$.
- Every value a from the interval $[\underline{a}, \overline{a}]$ is smaller than or equal to $\overline{b}$.
- If $\overline{a} \le \overline{b}$, this implies that for every value $a \le \overline{a}$, we have $a \le \overline{b}$;
- Vice versa, if *every* $a \in [\underline{a}, \overline{a}]$ satisfies the inequality $a \le \overline{b}$, then this inequality holds for $\overline{a} \in [\underline{a}, \overline{a}]$.
- Thus, the first formula is equivalent to $\overline{a} \le \overline{b}$.

47. Proof: Second Formula

$$\forall b \in \mathbf{b}\, \exists a \in \mathbf{a}\, (a \le b).$$

- If $a \le b$ for some a for which $\underline{a} \le a \le \overline{a}$ then, by transitivity, we get $\underline{a} \le b$.
- Vice versa, if $\underline{a} \le b$, then $a \le b$ for some $a \in [\underline{a}, \overline{a}]$: namely, for $a = \overline{a}$.
- Every value b from the interval $[\underline{b}, \overline{b}]$ is larger than or equal to $\underline{a}$.
- If $\underline{a} \le \underline{b}$, this implies that for every value $b \ge \underline{b}$, we have $\underline{a} \le b$.
- Vice versa, if *every* $b \in [\underline{b}, \overline{b}]$ satisfies the inequality $\underline{a} \le b$, then this inequality holds for $\underline{b} \in [\underline{b}, \overline{b}]$.
- Thus, the second formula is equivalent to $\underline{a} \le \underline{b}$.

48. Combining the two formulas: the resulting logical interpretation.

- The following two formulas together are equivalent to lattice order:
  - The first formula is equivalent to $\overline{a} \le \overline{b}$.
  - The second formula is equivalent to $\underline{a} \le \underline{b}$.
- Namely, the order $\underline{a} \le \overline{b}$ means that every element $a \in \mathbf{a}$ is smaller than or equal to every element $b \in \mathbf{b}$.
- In contrast, the lattice order is equivalent to the following two statements:
  - for a given value $a \in \mathbf{a}$, once we know this value, we can always select $b \in \mathbf{b}$ for which $a \le b$;
  - for a given value $b \in \mathbf{b}$, once we know this value, we can always select $a \in \mathbf{a}$ for which $a \le b$.

49. Possible generalizations of this interpretation

- If we consider intervals from the real line, the following relation forms a *lattice*:

$$[\underline{a}, \overline{a}] \le [\underline{b}, \overline{b}] \Leftrightarrow (\underline{a} \le \underline{b} \,\&\, \overline{a} \le \overline{b})$$

- For every two intervals, there is a least upper bound and a greatest lower bound.
- A similar definition can be formulated for a more general case of intervals over a partially ordered set:

$$[a, b] \stackrel{\text{def}}{=} \{x : a \preccurlyeq x \preccurlyeq b\}$$

- In this case, the above relation is no longer a lattice, but we can still prove that it is equivalent to:

$$\forall a \in \mathbf{a} \, \exists b \in \mathbf{b} \, (a \preccurlyeq b) \text{ and } \forall b \in \mathbf{b} \, \exists a \in \mathbf{a} \, (a \preccurlyeq b).$$

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53. Auxiliary Results: General Idea and First Example

- For each property of intervals in an ordered set A, we analyze:
  - which properties need to be satisfied for A_1 and A_2
  - so that the corresponding property is satisfies for intervals in $A_1 \times A_2$.
- *Connectedness property (CP):* for every two points $a, a' \in A$, there exists an interval that contains a and a':

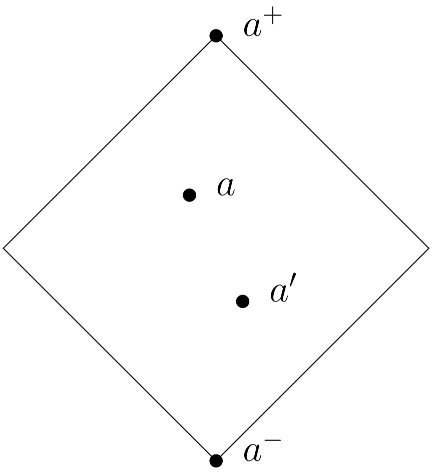
$$\forall a\, \forall a'\, \exists a^-\, \exists a^+\, (a^- \preccurlyeq a, a' \preccurlyeq a^+).$$

- This property is equivalent to two properties:
  - A is *upward-directed*: $\forall a\, \forall a'\, \exists a^+\, (a, a' \preccurlyeq a^+)$;
  - A is *downward-directed*: $\forall a\, \forall a'\, \exists a^-\, (a^- \preccurlyeq a, a')$.
- *Cartesian product*: A is upward-(downward-) directed $\Leftrightarrow$ both A_1 and A_2 are upward-(downward-) directed.

54. Connectedness Property Illustrated

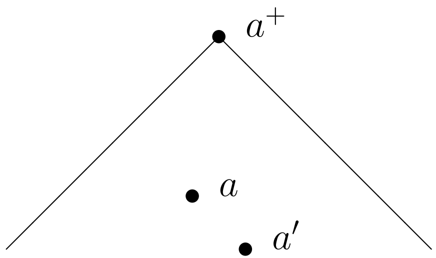
Connectedness property (CP): for every two points $a, a' \in A$, there exists an interval that contains a and a':

$$\forall a\, \forall a'\, \exists a^-\, \exists a^+\, (a^- \preccurlyeq a, a' \preccurlyeq a^+).$$

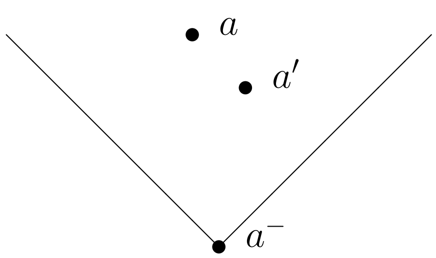


55. Upward and Downward Directed: Illustrated

Upward-directed: $\forall a\, \forall a'\, \exists a^+\, (a, a' \preccurlyeq a^+)$;



Downward-directed: $\forall a\, \forall a'\, \exists a^-\, (a^- \preccurlyeq a, a')$.



56. First Example, Case of Cartesian Product: Proof

- *Part 1:*
  - Let us assume that $A_1 \times A_2$ is upward-directed.
  - We want to prove that A_1 is upward-directed.
  - For any $a_1, a_1' \in A_1$, take any $a_2 \in A_2$, then
    $$\exists a^+ = (a_1^+, a_2^+) \text{ such that } (a_1, a_2), (a_1', a_2) \preccurlyeq a^+.$$
  - Hence $a_1, a_1' \preccurlyeq_1 a_1^+$, i.e., A_1 is upward-directed.
- *Part 2:*
  - Assume that both A_i are upward-directed.
  - We want to prove that $A_1 \times A_2$ is upward-directed.
  - For any $a = (a_1, a_2)$ and $a' = (a_1', a_2')$, for $i = 1, 2$,
    $$\exists\, a_i^+\, (a_i, a_i' \preccurlyeq_i a_i^+).$$
  - Hence $(a_1, a_2), (a_1', a_2') \preccurlyeq (a_1^+, a_2^+)$, i.e., $A_1 \times A_2$ is upward-directed.

57. First Example: Case of Lexicographic Product

- $A_1 \times A_2$ is upward-directed $\Leftrightarrow$ the following two conditions hold:
  - the set A_1 is upward-directed, and
  - if A_1 has a maximal element $\overline{a}_1$ (= for which there are no a_1 with $\overline{a}_1 \prec_1 a_1$), then A_2 is upward-directed.
- $A_1 \times A_2$ is downward-directed $\Leftrightarrow$ the following two conditions hold:
  - the set A_1 is downward-directed, and
  - if A_1 has a minimal element $\underline{a}_1$ (= for which there are no a_1 for which $a_1 \prec_1 \underline{a}_1$), then A_2 is downward-directed.

58. Case of Lexicographic Product: Proof

- Let us assume that $A_1 \times A_2$ is upward-directed.
- *Part 1:*
  - We want to prove that A_1 is upward-directed.
  - For any $a_1, a_1' \in A_1$, take any $a_2 \in A_2$, then

$$\exists a^+ = (a_1^+, a_2^+) \text{ for which } (a_1, a_2), (a_1', a_2) \preccurlyeq a^+.$$

  - Hence $a_1, a_1' \preccurlyeq_1 a_1^+$, i.e., A_1 is upward-directed.
- *Part 2:*
  - Let $\overline{a}_1$ be a maximal element of A_1.
  - For any $a_2, a_2' \in A_2$, we have

$$\exists a^+ = (a_1^+, a_2^+) \text{ for which } (\overline{a}_1, a_2), (\overline{a}_1, a_2') \preccurlyeq a^+.$$

  - Here, $\overline{a}_1 \preccurlyeq_1 a_1^+$ and since $\overline{a}_1$ is maximal, $a_1^+ = \overline{a}_1$.
  - Hence $a_2, a_2' \preccurlyeq_2 a_2^+$, i.e., A_2 is upward-directed.

59. Proof (cont-d)

- Let us assume that A_1 is upward-directed.
- Let us assume that if A_1 has a maximal element, then A_2 is upward-directed.
- We want to prove that $A_1 \times A_2$ is upward-directed.
- Take any $a = (a_1, a_2)$ and $a' = (a'_1, a'_2)$ from $A_1 \times A_2$.
- Since A_1 is upward-directed, $\exists a_1^+ \, (a_1, a'_1 \preccurlyeq_1 a_1^+)$.
- If $a_1 \prec_1 a_1^+$, then $(a_1, a_2), (a'_1, a'_2) \preccurlyeq (a_1^+, a'_2)$.
- If $a'_1 \prec_1 a_1^+$, then $(a_1, a_2), (a'_1, a'_2) \preccurlyeq (a_1^+, a_2)$.
- If $a_1 = a_1^+ = a'_1$, and a_1 is not a maximal element, then $\exists a''_1 \, (a_1 \prec_1 a''_1)$, hence $(a_1, a_2), (a'_1, a'_2) \preccurlyeq (a''_1, s_2)$.
- If $a_1 = a_1^+ = a'_1$, and a_1 is a maximal element, then A_2 is upward-directed, hence $\exists a_2^+ \, (a_2, a'_2 \preccurlyeq_2 a_2^+)$ and

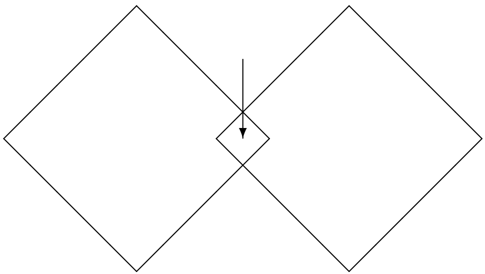
$$(a_1, a_2), (a_1, a'_2) \preccurlyeq (a_1, a_2^+).$$

60. Second Example: Intersection Property

- The intersection of every two intervals is an interval.
- *Comment:* this is true for intervals on the real line.
- This can be similarly reduced to two properties:
  - the intersection of every two future cones $C_a^+ \stackrel{\text{def}}{=} \{b : a \preccurlyeq b\}$ is a future cone;
  - the intersection of every two past cones $C_a^- \stackrel{\text{def}}{=} \{b : b \preccurlyeq a\}$ is a past cone.
- If both properties hold, then the intersection of every two intervals $[a, b] = C_a^+ \cap C_b^-$ is an interval.
- Ordered sets with such C^+ and C^- properties are called *upper* and *lower* semi-lattices.
- *For Cartesian product:* $A_1 \times A_2$ is an upper (lower) semi-lattice $\Leftrightarrow$ both A_i are upper (lower) semi-lattices.

61. Intersection Property Illustrated

Intersection property for intervals:



Upper and lower semi-lattice properties: