Regular completions

John Harding, Qin Yang

Abstract

Let P be a partially ordered set. If each $x, y \in P$ have a least upper bound $x \vee y$ and greatest lower bound $x \wedge y$, then P is called a lattice. If the least upper bound $\bigvee S$ and greatest lower bound $\bigwedge S$ exist for all $S \subseteq P$, then P is called a complete lattice. A completion of a lattice L is a pair (α, C) where C is a complete lattice and $\alpha : L \to C$ is a lattice embedding. A completion $\alpha : L \to C$ is said to be regular if for any $S \subseteq L$ if $\bigvee S$ exists in L then $\alpha(\bigvee S) = \bigvee \alpha(S)$ and if $\bigwedge S$ exists in L then $\alpha(\bigwedge S) = \bigwedge \alpha(S)$, i.e. α preserves all existing joins and meets. A completion $\alpha : L \to C$ is said to be meet dense if for each $c \in C$ we have $c = \bigwedge\{\alpha(a) \mid a \in L$ and $\alpha(a) \geq c\}$. Join dense is similar.

MacNeile completions are regular and both join dense and meet dense. My advisor Dr. John harding has proved that only two varieties of lattices (the trival variety and the variety of all lattices) are closed (i.e. if $L \in V$, then $Mc(L) \in V$) under MacNeile completions.

The question is "Are there any varieties of lattices that admit regular completions besides the trival variety and the all lattices variety)? ". This is an open problem.

Recently, my advisor Dr. John Harding and I got a result: If V is a variety of lattices that admits a meet-dense and regular completion (for each $L \in V$ there is a completion $\alpha : L \to C$ with $C \in V$ that is meet-dense and regular), then V is either the trival variety of one element lattices or the variety of all lattices.