Why Boxes for Multi-D Uncertainty?

Christian Servin[1], Erick Duarte[2], Francisco Rodriguez[2],
Olga Kosheleva[2], and Vladik Kreinovich[2]
[1]El Paso Community College, 919 Hunter
El Paso, TX 79915 USA, cservin@gmail.com
[2]University of Texas at El Paso, El Paso, TX 79968 USA
{ejduarte2,farodriguezcabral}@miners.utep.edu
olgak@utep.edu, vladik@utep.edu

Formulation of the problem. In practice, we often do not have a complete information about the values of several quantities $x_1, \ldots, x_n$: several possible tuples $x = (x_1, \ldots, x_n)$ are consistent with our knowledge. How can we describe the set of possible values of x? In many practical applications, approximating this set by a box $[\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_n, \overline{x}_n]$ leads to reasonable estimates. Why boxes? Why not other families of sets?

What are reasonable properties of the corresponding family F of approximating sets? Usually, we know some bounds on all x_i, so each set S should be bounded.

It is reasonable to require that each set $S \in F$ is closed: indeed, if $a = \lim a_n$ for $a_n \in S$, then for any measurement accuracy, a is indistinguishable from some some possible $a_n \in S$ and thus, we will never be able to tell that a is not possible.

Similarly, the family F should be closed. Also, often, the observed tuple consists of two independent components $x_i = y_i + z_i$; so, if the set Y of all y's and the set Z of all z's are possible, then the *Minkowski sum* $Y + Z \stackrel{\text{def}}{=} \{y + z : y \in Y, z \in Z\}$ is also possible. So, F should be closed under Minkowski sum.

Finally, the numerical values of each quantity x_i change if we change the starting point and/or measuring unit: $x_i \to a_i + b_i \cdot x_i$. These changes do not change what is possible and what is not, so the family F should be closed under these transformations.

Our result: Every family F that satisfies these properties contains all the boxes. Thus, the boxes form the simplest possible approximating family.

Idea of the proof. For each set S, we can take $S_1 = S$, $S_{k+1} = 0.5 \cdot (S_k + S_k)$, etc. In the limit, we get a convex hull of S. By appropriate re-scaling $x_i \to b_i \cdot x_i$, we can shrink this set S in all directions except one i_0. In the limit, we get an interval parallel to the i_0-th axis. By shifting and re-scaling, we get all possible intervals parallel to this axis. For n intervals $[\underline{x}_i, \overline{x}_i]$ each of which is parallel to the i-th axis and has all other coordinates 0s, their Minkowski sum is the box $[\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_n, \overline{x}_n]$. Thus, each family F satisfying the above properties contains all the boxes. (Also, the class of all boxes satisfies these properties.)