Commonsense “And”-Operations

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Why “and”-operations. In many practical applications, a certain effect appears if several conditions $C_1, C_2, \ldots$ are satisfied. For each of these conditions C_i, we can elicit, from the experts, the degree $d_i \in [0,1]$ to which this condition is satisfied. However, there are many possible conditions, and it is not possible to extract, from the experts, a degree to which each possible “and”-combination $C_1 \,\&\, C_2 \,\&\, \ldots$ is satisfied. Thus, we need to be able, given degrees of confidence a and b in statements A and B, to estimate the degree to which the “and”-combination $A \,\&\, B$ is satisfied. This estimate is denoted by $f_\&(a,b)$, and the algorithm for computing this estimate is known as an “and”-operation or, for historical reason, a t-norm.

How usual “and”-operations are obtained. It is usually assumed that in situations when about each of the combined statements, we are absolute certain either that this statement is true or that this statement is false, the “and”-operation should return the true value of the corresponding “and”-statement, i.e., that we should have $f_\&(0,0) = f_\&(0,1) = f_\&(1,0) = 0$ and $f_\&(1,1) = 1$.

To extend these values to all possible combinations of $a \in [0,1]$ and $b \in [0,1]$, a reasonable idea is to use linear interpolation over each variable, i.e., to assume that for every a, the mapping $b \mapsto f_\&(a,b)$ is linear, and for every b, the mapping $a \mapsto f_\&(a,b)$ is linear. As a result, we conclude that the desired function is bilinear, i.e., has the form $f_\&(a,b) = c_0 + c_a \cdot a + c_b \cdot b + c_{ab} \cdot a \cdot b$ for some coefficients c_i. Taking into account the above conditions for $a, b \in \{0,1\}$, we conclude that $f_\&(a,b) = a \cdot b$. This is indeed one of the most frequently used “and”-operations.

Comment. Similarly, linear interpolation enables us to similarly determine that an appropriate “or”-operation (historically also known as t-conorm) has the form $f_\vee(a,b) = a + b - a \cdot b$.

Need to go beyond the usual “and”-operations. In some cases, when we say “and”, we mean exactly the logical “and”: all conditions must be absolutely satisfied. However, in many practical problems, “and” is “softer” than that. For example, if you ask a person who is planning to buy a house what house he/she wants, the person will say: not too far away *and* spacey *and* not very expensive *and* reasonably well thermo-isolated *and* in a nice neighborhood, etc. However, this “and” does not mean literal “and”: if this person finds a house that satisfied most of these conditions, he/she will gladly buy it. How can we describe such commonsense “and”-operations?

Our solution. In this talk, we consider the case when we only have two conditions. For a commonsense “and”-operation $F_\&(a,b)$, it is reasonable to still have $F_\&(0,0) = 0$ and $F_\&(1,1) = 1$, but if only one of the conditions A and B is satisfied, then the statement $A \,\&\, B$ should also be to some extent true. In other words, we should have $F_\&(0,1) = F_\&(1,0) = \alpha$ for some small $\alpha > 0$. In this case, we get $F_\&(a,b) = \alpha \cdot (a+b) + (1-2\alpha) \cdot a \cdot b$, i.e., equivalently,

$$F_\&(a,b) = (1-\alpha) \cdot a \cdot b + \alpha \cdot (a + b - a \cdot b) = (1-\alpha) \cdot f_\&(a,b) + \alpha \cdot f_\vee(a,b).$$

In other words, this operation is a convex combination of the usual “and”- and “or”-operations.

Comment. In contrast to the usual “and”- and “or”-operations, the commonsense “and”-operation is not associative. Thus, while we can define $f_\&(a,b,c)$ as, e.g., $f_\&(a, f_\&(b,c))$ or as $f_\&(f_\&(a,b),c)$ – and the result will not change, with the commonsense “and”-operation, we will have two different results. So, e.g., for three inputs, we get a more general formula

$$F_\&(a,b,c) = \alpha \cdot (a+b+c) + \beta \cdot (a \cdot b + b \cdot c + a \cdot c) + (1 - 3\alpha - 3\beta) \cdot a \cdot b \cdot c.$$